Mathematics for College Liberal Arts B.E.S.T. Instructional Guide for Mathematics

The B.E.S.T. Instructional Guide for Mathematics (B1G-M) is intended to assist educators with planning for student learning and instruction aligned to Florida's Benchmarks for Excellent Student Thinking (B.E.S.T.) Standards. This guide is designed to aid high-quality instruction through the identification of components that support the learning and teaching of the B.E.S.T. Mathematics Standards and Benchmarks. The B1G-M includes an analysis of information related to the B.E.S.T. Standards for Mathematics within this specific mathematics course, the instructional emphasis and aligned resources. This document is posted on the B.E.S.T. Standards for Mathematics webpage of the Florida Department of Education's website and will continue to undergo edits as needed.

Structural Framework and Intentional Design of the B.E.S.T. Standards for Mathematics

Florida's B.E.S.T. Standards for Mathematics were built on the following.

- The coding scheme for the standards and benchmarks was changed to be consistent with other content areas. The new coding scheme is structured as follows: Content.GradeLevel.Strand.Standard.Benchmark.
- Strands were streamlined to be more consistent throughout.
- The standards and benchmarks were written to be clear and concise to ensure that they are easily understood by all stakeholders.
- The benchmarks were written to allow teachers to meet students' individual skills, knowledge and ability.
- The benchmarks were written to allow students the flexibility to solve problems using a method or strategy that is accurate, generalizable and efficient depending on the content (i.e., the numbers, expressions or equations).
- The benchmarks were written to allow for student discovery (i.e., exploring) of strategies rather than the teaching, naming and assessing of each strategy individually.
- The benchmarks were written to support multiple pathways for success in career and college for students.
- The benchmarks should not be taught in isolation but should be combined purposefully.
- The benchmarks may be addressed at multiple points throughout the year, with the intention of gaining mastery by the end of the year.
- Appropriate progression of content within and across strands was developed for each grade level and across grade levels.
- There is an intentional balance of conceptual understanding and procedural fluency with the application of accurate real-world context intertwined within mathematical concepts for relevance.
- The use of other content areas, like science and the arts, within real-world problems should be accurate, relevant, authentic and reflect grade-level appropriateness.

Components of the B.E.S.T. Instructional Guide for Mathematics

The following table is an example of the layout for each benchmark and includes the defining attributes for each component. It is important to note that instruction should not be limited to the possible connecting benchmarks, related terms, strategies or examples provided. To do so would strip the intention of an educator meeting students' individual skills, knowledge and abilities.

Benchmark
focal point for instruction within lesson or task

This section includes the benchmark as identified in the B.E.S.T. Standards for Mathematics. The benchmark, also referred to as the Benchmark of Focus, is the focal point for student learning and instruction. The benchmark, and its related example(s) and clarification(s), can also be found in the course description. The 9-12 benchmarks may be included in multiple courses; select the example(s) or clarification(s) as appropriate for the identified course.

Connecting Benchmarks/Horizontal Alignment
in other standards within the grade level or course

This section includes a list of connecting benchmarks that relate horizontally to the Benchmark of Focus. Horizontal alignment is the intentional progression of content within a grade level or course linking skills within and across strands. Connecting benchmarks are benchmarks that either make a mathematical connection or include prerequisite skills. The information included in this section is not a comprehensive list, and educators are encouraged to find other connecting benchmarks. Additionally, this list will not include benchmarks from the same standard since benchmarks within the same standard already have an inherent connection.

Terms from the K-12 Glossary

This section includes terms from Appendix C: K-12 Glossary, found within the B.E.S.T. Standards for Mathematics document, which are relevant to the identified Benchmark of Focus. The terms included in this section should not be viewed as a comprehensive vocabulary list, but instead should be considered during instruction or act as a reference for educators.

Vertical Alignment
across grade levels or courses

This section includes a list of related benchmarks that connect vertically to the Benchmark of Focus. Vertical alignment is the intentional progression of content from one year to the next, spanning across multiple grade levels. Benchmarks listed in this section make mathematical connections from prior grade levels or courses in future grade levels or courses within and across strands. If the Benchmark of Focus is a new concept or skill, it may not have any previous benchmarks listed. Likewise, if the Benchmark of Focus is a mathematical skill or concept that is finalized in learning and does not have any direct connection to future grade levels or courses, it may not have any future benchmarks listed. The information included in this section is not a comprehensive list, and educators are encouraged to find other benchmarks within a vertical progression.

Purpose and Instructional Strategies

This section includes further narrative for instruction of the benchmark and vertical alignment. Additionally, this section may also include the following:

- explanations and details for the benchmark;
- vocabulary not provided within Appendix C;
- possible instructional strategies and teaching methods; and
- strategies to embed potentially related Mathematical Thinking and Reasoning Standards (MTRs).

Common Misconceptions or Errors

This section will include common student misconceptions or errors and may include strategies to address the identified misconception or error. Recognition of these misconceptions and errors enables educators to identify them in the classroom and make efforts to correct the misconception or error. This corrective effort in the classroom can also be a form of formative assessment within instruction.

Instructional Tasks

demonstrate the depth of the benchmark and the connection to the related benchmarks

This section will include example instructional tasks, which may be open-ended and are intended to demonstrate the depth of the benchmark. Some instructional tasks include integration of the Mathematical Thinking and Reasoning Standards (MTRs) and related benchmark(s). Enrichment tasks may be included to make connections to benchmarks in later grade levels or courses. Tasks may require extended time, additional materials and collaboration.

Instructional Items

demonstrate the focus of the benchmark

This section will include example instructional items which may be used as evidence to demonstrate the students' understanding of the benchmark. Items may highlight one or more parts of the benchmark.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

Mathematical Thinking and Reasoning Standards

MTRs: Because Math Matters

Florida students are expected to engage with mathematics through the Mathematical Thinking and Reasoning Standards (MTRs) by utilizing their language as a self-monitoring tool in the classroom, promoting deeper learning and understanding of mathematics. The MTRs are standards which should be used as a lens when planning for student learning and instruction of the B.E.S.T. Standards for Mathematics.

Structural Framework and Intentional Design of the Mathematical Thinking and Reasoning Standards

The Mathematical Thinking and Reasoning Standards (MTRs) are built on the following.

- The MTRs have the same coding scheme as the standards and benchmarks; however, they are written at the standard level because there are no benchmarks.
- In order to fulfill Florida's unique coding scheme, the 5th place (benchmark) will always be a "1" for the MTRs.
- The B.E.S.T. Standards for Mathematics should be taught through the lens of the MTRs.
- At least one of the MTRs should be authentically and appropriately embedded throughout every lesson based on the expectation of the benchmark(s).
- The bulleted language of the MTRs were written for students to use as self-monitoring tools during daily instruction.
- The clarifications of the MTRs were written for teachers to use as a guide to inform their instructional practices.
- The MTRs ensure that students stay engaged, persevere in tasks, share their thinking, balance conceptual understanding and procedures, assess their solutions, make connections to previous learning and extended knowledge, and apply mathematical concepts to real-world applications.
- The MTRs should not stand alone as a separate focus for instruction, but should be combined purposefully.
- The MTRs will be addressed at multiple points throughout the year, with the intention of gaining mastery of mathematical skills by the end of the year and building upon these skills as they continue in their K-12 education.

MA.K12.MTR.1.1 Actively participate in effortful learning both individually and collectively.

Mathematicians who participate in effortful learning both individually and with others:

- Analyze the problem in a way that makes sense given the task.
- Ask questions that will help with solving the task.
- Build perseverance by modifying methods as needed while solving a challenging task.
- Stay engaged and maintain a positive mindset when working to solve tasks.
- Help and support each other when attempting a new method or approach.

Clarifications:
Teachers who encourage students to participate actively in effortful learning both individually and with others:

- Cultivate a community of growth mindset learners.
- Foster perseverance in students by choosing tasks that are challenging.
- Develop students' ability to analyze and problem solve.
- Recognize students' effort when solving challenging problems.

MA.K12.MTR.2.1 Demonstrate understanding by representing problems in multiple ways.

Mathematicians who demonstrate understanding by representing problems in multiple ways:

- Build understanding through modeling and using manipulatives.
- Represent solutions to problems in multiple ways using objects, drawings, tables, graphs and equations.
- Progress from modeling problems with objects and drawings to using algorithms and equations.
- Express connections between concepts and representations.
- Choose a representation based on the given context or purpose.

Clarifications:
Teachers who encourage students to demonstrate understanding by representing problems in multiple ways:

- Help students make connections between concepts and representations.
- Provide opportunities for students to use manipulatives when investigating concepts.
- Guide students from concrete to pictorial to abstract representations as understanding progresses.
- Show students that various representations can have different purposes and can be useful in different situations.

MA.K12.MTR.3.1 Complete tasks with mathematical fluency.

Mathematicians who complete tasks with mathematical fluency:

- Select efficient and appropriate methods for solving problems within the given context.
- Maintain flexibility and accuracy while performing procedures and mental calculations.
- Complete tasks accurately and with confidence.
- Adapt procedures to apply them to a new context.
- Use feedback to improve efficiency when performing calculations.

Clarifications:
Teachers who encourage students to complete tasks with mathematical fluency:

- Provide students with the flexibility to solve problems by selecting a procedure that allows them to solve efficiently and accurately.
- Offer multiple opportunities for students to practice efficient and generalizable methods.
- Provide opportunities for students to reflect on the method they used and determine if a more efficient method could have been used.

MA.K12.MTR.4.1 Engage in discussions that reflect on the mathematical thinking of self and others.

Mathematicians who engage in discussions that reflect on the mathematical thinking of self and others:

- Communicate mathematical ideas, vocabulary and methods effectively.
- Analyze the mathematical thinking of others.
- Compare the efficiency of a method to those expressed by others.
- Recognize errors and suggest how to correctly solve the task.
- Justify results by explaining methods and processes.
- Construct possible arguments based on evidence.

Clarifications:
Teachers who encourage students to engage in discussions that reflect on the mathematical thinking of self and others:

- Establish a culture in which students ask questions of the teacher and their peers, and error is an opportunity for learning.
- Create opportunities for students to discuss their thinking with peers.
- Select, sequence and present student work to advance and deepen understanding of correct and increasingly efficient methods.
- Develop students' ability to justify methods and compare their responses to the responses of their peers.

MA.K12.MTR.5.1 Use patterns and structure to help understand and connect mathematical concepts.

Mathematicians who use patterns and structure to help understand and connect mathematical concepts:

- Focus on relevant details within a problem.
- Create plans and procedures to logically order events, steps or ideas to solve problems.
- Decompose a complex problem into manageable parts.
- Relate previously learned concepts to new concepts.
- Look for similarities among problems.
- Connect solutions of problems to more complicated large-scale situations.

Clarifications:
Teachers who encourage students to use patterns and structure to help understand and connect mathematical concepts:

- Help students recognize the patterns in the world around them and connect these patterns to mathematical concepts.
- Support students to develop generalizations based on the similarities found among problems.
- Provide opportunities for students to create plans and procedures to solve problems.
- Develop students' ability to construct relationships between their current understanding and more sophisticated ways of thinking.

MA.K12.MTR.6.1 Assess the reasonableness of solutions.

Mathematicians who assess the reasonableness of solutions:

- Estimate to discover possible solutions.
- Use benchmark quantities to determine if a solution makes sense.
- Check calculations when solving problems.
- Verify possible solutions by explaining the methods used.
- Evaluate results based on the given context.

Clarifications:
Teachers who encourage students to assess the reasonableness of solutions:

- Have students estimate or predict solutions prior to solving.
- Prompt students to continually ask, "Does this solution make sense? How do you know?"
- Reinforce that students check their work as they progress within and after a task.
- Strengthen students' ability to verify solutions through justifications.

MA.K12.MTR.7.1 Apply mathematics to real-world contexts.

Mathematicians who apply mathematics to real-world contexts:

- Connect mathematical concepts to everyday experiences.
- Use models and methods to understand, represent and solve problems.
- Perform investigations to gather data or determine if a method is appropriate.
- Redesign models and methods to improve accuracy or efficiency.

Clarifications:
Teachers who encourage students to apply mathematics to real-world contexts:

- Provide opportunities for students to create models, both concrete and abstract, and perform investigations.
- Challenge students to question the accuracy of their models and methods.
- Support students as they validate conclusions by comparing them to the given situation.
- Indicate how various concepts can be applied to other disciplines.

Examples of Teacher and Student Moves for the MTRs

Below are examples that demonstrate the embedding of the MTRs within the mathematics classroom. The provided teacher and student moves are examples of how some MTRs could be incorporated into student learning and instruction keeping in mind the benchmark(s) that are the focal point of the lesson or task. The information included in this table is not a comprehensive list, and educators are encouraged to incorporate other teacher and student moves that support the MTRs.

MTR	**Student Moves**	**Teacher Moves**
MA.K12.MTR.1.1 *Actively participate in effortful learning both individually and collectively.*	• Students engage in the task through individual analysis, student-to-teacher interaction and student-to-student interaction. • Students ask task-appropriate questions to self, the teacher and to other students. *(MTR.4.1)* • Students have a positive productive struggle exhibiting growth mindset, even when making a mistake. • Students stay engaged in the task to a purposeful conclusion while modifying methods, when necessary, in solving a problem through self-analysis and perseverance.	• Teacher provides flexible options (i.e., differentiated, challenging tasks that allow students to actively pursue a solution both individually and in groups) so that all students have the opportunity to access and engage with instruction, as well as demonstrate their learning. • Teacher creates a physical environment that supports a growth mindset and will ensure positive student engagement and collaboration. • Teacher provides constructive, encouraging feedback to students that recognizes their efforts and the value of analysis and revision. • Teacher provides appropriate time for student processing, productive struggle and reflection. • Teacher uses data and questions to focus students on their thinking; help students determine their sources of struggle and to build understanding. • Teacher encourages students to ask appropriate questions of other students and of the teacher including questions that examine accuracy. *(MTR.4.1)*

MTR	Student Moves	Teacher Moves
MA.K12.MTR.2.1 *Demonstrate understanding by representing problems in multiple ways.*	• Students represent problems concretely using objects, models and manipulatives. • Students represent problems pictorially using drawings, models, tables and graphs. • Students represent problems abstractly using numerical or algebraic expressions and equations. • Students make connections and select among different representations and methods for the same problem, as appropriate to different situations or context. *(MTR.3.1)*	• Teacher provides students with objects, models, manipulatives, appropriate technology and real-world situations. *(MTR.7.1)* • Teacher encourages students to use drawings, models, tables, expressions, equations and graphs to represent problems and solutions. • Teacher questions students about making connections between different representations and methods and challenges students to choose one that is most appropriate to the context. *(MTR.3.1)* • Teacher encourages students to explain their different representations and methods to each other. *(MTR.4.1)* • Teacher provides opportunities for students to choose appropriate methods and to use mathematical technology.
MA.K12.MTR.3.1 *Complete tasks with mathematical fluency.*	• Students complete tasks with flexibility, efficiency and accuracy. • Students use feedback from peers and teachers to reflect on and revise methods used. • Students build confidence through practice in a variety of contexts and problems. *(MTR.1.1)*	• Teacher provides tasks and opportunities to explore and share different methods to solve problems. *(MTR.1.1)* • Teacher provides opportunities for students to choose methods and reflect (i.e., through error analysis, revision, summarizing methods or writing) on the efficiency and accuracy of the method(s) chosen. • Teacher asks questions and gives feedback to focus student thinking to build efficiency of accurate methods. • Teacher offers multiple opportunities to practice generalizable methods.

MTR	Student Moves	Teacher Moves
MA.K12.MTR.4.1 *Engage in discussions that reflect on the mathematical thinking of self and others.*	• Students use content specific language to communicate and justify mathematical ideas and chosen methods. • Students use discussions and reflections to recognize errors and revise their thinking. • Students use discussions to analyze the mathematical thinking of others. • Students identify errors within their own work and then determine possible reasons and potential corrections. • When working in small groups, students recognize errors of their peers and offers suggestions.	• Teacher provides students with opportunities (through open-ended tasks, questions and class structure) to make sense of their thinking. *(MTR.1.1)* • Teacher uses precise mathematical language, both written and abstract, and encourages students to revise their language through discussion. • Teacher creates opportunities for students to discuss and reflect on their choice of methods, their errors and revisions and their justifications. • Teachers select, sequence and present student work to elicit discussion about different methods and representations. *(MTR.2.1, MTR.3.1)*

MTR	Student Moves	Teacher Moves
MA.K12.MTR.5.1 *Use patterns and structure to help understand and connect mathematical concepts.*	• Students identify relevant details in a problem in order to create plans and decompose problems into manageable parts. • Students find similarities and common structures, or patterns, between problems in order to solve related and more complex problems using prior knowledge.	• Teacher asks questions to help students construct relationships between familiar and unfamiliar problems and to transfer this relationship to solve other problems. *(MTR.1.1)* • Teacher provides students opportunities to connect prior and current understanding to new concepts. • Teacher provides opportunities for students to discuss and develop generalizations about a mathematical concept. *(MTR.3.1, MTR.4.1)* • Teacher allows students to develop an appropriate sequence of steps in solving problems. • Teacher provides opportunities for students to reflect during problem solving to make connections to problems in other contexts, noticing structure and making improvements to their process.
MA.K12.MTR.6.1 *Assess the reasonableness of solutions.*	• Students estimate a solution, including using benchmark quantities in place of the original numbers in a problem. • Students monitor calculations, procedures and intermediate results during the process of solving problems. • Students verify and check if solutions are viable, or reasonable, within the context or situation. *(MTR.7.1)* • Students reflect on the accuracy of their estimations and their solutions.	• Teacher provides opportunities for students to estimate or predict solutions prior to solving. • Teacher encourages students to compare results to estimations and revise if necessary for future situations. *(MTR.5.1)* • Teacher prompts students to self-monitor by continually asking, "Does this solution or intermediate result make sense? How do you know?" • Teacher encourages students to provide explanations and justifications for results to self and others. *(MTR.4.1)*

MTR	Student Moves	Teacher Moves
MA.K12.MTR.7.1 *Apply mathematics to real-world contexts.*	• Students connect mathematical concepts to everyday experiences. • Students use mathematical models and methods to understand, represent and solve real-world problems. • Students investigate, research and gather data to determine if a mathematical model is appropriate for a given situation from the world around them. • Students re-design models and methods to improve accuracy or efficiency.	• Teacher provides real-world context to help students build understanding of abstract mathematical ideas. • Teacher encourages students to assess the validity and accuracy of mathematical models and situations in real-world context, and to revise those models if necessary. • Teacher provides opportunities for students to investigate, research and gather data to determine if a mathematical model is appropriate for a given situation from the world around them. • Teacher provides opportunities for students to apply concepts to other content areas.

Mathematics for College Liberal Arts Areas of Emphasis

In Mathematics for College Liberal Arts, instructional time will emphasize five areas:

(1) analyzing and applying linear and exponential functions within a real-world context;
(2) utilizing geometric concepts to solve real-world problems;
(3) extending understanding of probability theory;
(4) representing and interpreting univariate and bivariate data; and
(5) developing understanding of logic and set theory.

The purpose of the areas of emphasis is not to guide specific units of learning and instruction, but rather provide insight on major mathematical topics that will be covered within this mathematics course. In addition to its purpose, the areas of emphasis are built on the following.

- Supports the intentional horizontal progression within the strands and across the strands in this grade level or course.
- Student learning and instruction should not focus on the stated areas of emphasis as individual units.
- Areas of emphasis are addressed within standards and benchmarks throughout the course so that students are making connections throughout the school year.
- Some benchmarks can be organized within more than one area.
- Supports the communication of the major mathematical topics to all stakeholders.
- Benchmarks within the areas of emphasis should not be taught within the order in which they appear. To do so would strip the progression of mathematical ideas and miss the opportunity to enhance horizontal progressions within the grade level or course.

The table on the next page shows how the benchmarks within this mathematics course are embedded within the areas of emphasis.

		Linear and Exponential Functions	Geometric Concepts	Probability Theory	Univariate and Bivariate Data	Logic and Set Theory
Algebraic Reasoning	MA.912.AR.2.5	x			x	
	MA.912.AR.5.3	x			x	
	MA.912.AR.5.4	x			x	
	MA.912.AR.5.5	x			x	
	MA.912.AR.5.6	x			x	
Functions	MA.912.F.1.6	x			x	
	MA.912.F.1.8	x			x	
Financial Literacy	MA.912.FL.3.1	x			x	
	MA.912.FL.3.2	x			x	
	MA.912.FL.3.4	x			x	
Geometric Reasoning	MA.912.GR.1.6		x			
	MA.912.GR.2.4		x			
	MA.912.GR.4.3		x			
	MA.912.GR.4.4		x			
	MA.912.GR.4.5		x			
	MA.912.GR.4.6		x			
Trigonometry	MA.912.T.1.2		x			
Data Analysis & Probability	MA.912.DP.1.1				x	
	MA.912.DP.1.2				x	
	MA.912.DP.2.1				x	
	MA.912.DP.2.4	x			x	
	MA.912.DP.2.9	x			x	
	MA.912.DP.4.1			x		
	MA.912.DP.4.2			x		
	MA.912.DP.4.3			x		
	MA.912.DP.4.4			x		
	MA.912.DP.4.5			x	x	
	MA.912.DP.4.6			x		
	MA.912.DP.4.7			x		
	MA.912.DP.4.8			x		
	MA.912.DP.4.9			x		
	MA.912.DP.4.10			x		

		Linear and Exponential Functions	Geometric Concepts	Probability Theory	Univariate and Bivariate Data	Logic and Set Theory
Logic and Discrete Theory	MA.912.LT.4.1					x
	MA.912.LT.4.2					x
	MA.912.LT.4.3					x
	MA.912.LT.4.4					x
	MA.912.LT.4.5					x
	MA.912.LT.4.9					x
	MA.912.LT.4.10					x
	MA.912.LT.5.1					x
	MA.912.LT.5.4					x
	MA.912.LT.5.5					x
	MA.912.LT.5.6					x

MA.912.AR.2 *Write, solve and graph linear equations, functions and inequalities in one and two variables.*

MA.912.AR.2.5

Benchmark

MA.912.AR.2.5 Solve and graph mathematical and real-world problems that are modeled with linear functions. Interpret key features and determine constraints in terms of the context.

Algebra I Example: Lizzy's mother uses the function $C(p) = 450 + 7.75p$, where $C(p)$ represents the total cost of a rental space and p is the number of people attending, to help budget Lizzy's 16th birthday party. Lizzy's mom wants to spend no more than $850 for the party. Graph the function in terms of the context.

Benchmark Clarifications:
Clarification 1: Key features are limited to domain, range, intercepts and rate of change.
Clarification 2: Instruction includes the use of standard form, slope-intercept form and point-slope form.
Clarification 3: Instruction includes representing the domain, range and constraints with inequality notation, interval notation or set-builder notation.
Clarification 4: Within the Algebra I course, notations for domain and range are limited to inequality and set-builder.
Clarification 5: Within the Mathematics for Data and Financial Literacy course, problem types focus on money and business.

Connecting Benchmarks/Horizontal Alignment

- MA.912.F.1.6, MA.912.F.1.8,
- MA.912.FL.3.1, MA.912.FL.3.2, MA.912.FL.3.4
- MA.912.DP.2.4

Terms from the K-12 Glossary

- Coordinate Plane
- Domain
- Function notation
- Range
- Rate of change
- Slope
- x-intercept
- y-intercept

Vertical Alignment

Previous Benchmarks

- MA.8.AR.3.4
- MA.8.AR.3.5

Next Benchmarks

- MA.912.NSO.4.2
- MA.912.AR.9.8, MA.912.AR.9.9, MA.912.AR.9.10

Purpose and Instructional Strategies

In Algebra I, students solved real-world problems modeled with linear functions when given equations in all forms, as well as tables and written descriptions, and they determined and interpreted the domain, range and other key features. Students additionally interpreted key features and identified constraints. In Math for College Liberal Arts, students continue this work with a focus on real-world problems modeled by linear functions. In other courses, students will graph and solve problems involving linear programming, systems of equations in three variables and piecewise functions.

- Instruction in Algebra I included representing domain, range and constraints using words, inequality notation and set-builder notation. In Math for College Liberal Arts, instruction will also include interval notation.
    - Words
      If the domain is all real numbers, it can be written as "all real numbers" or "any value of x, such that x is a real number."
    - Inequality notation
      If the domain is all values of x greater than 2, it can be represented as $x > 2$.
    - Set-builder notation
      If the range is all values of y less than or equal to zero, it can be represented as $\{y|y \leq 0\}$ and is read as "all values of y such that y is less than or equal to zero."
    - Interval notation
      If the domain is all values of x less than or equal to 3, it can be represented as $(-\infty, 3]$. If the domain is all values of x greater than 3, it can be represented as $(3, \infty)$. If the range is all values greater than or equal to -1 but less than 5, it can be represented as $[-1, 5)$.
- Instruction includes the use of x-y notation and function notation.
- Instruction includes the connection between the independent variable and domain as well as the dependent variable and range.
- Instruction features a variety of real-world contexts.
- Students should have an understanding that linear graphs, without context, have no constraints on their domain and range. When specific contexts are modeled by linear functions, parts of the domain and range may not make sense and need to be removed, creating the need for constraints.
- Instruction includes the understanding that a real-world context can be represented by a linear two-variable equation even though it only has meaning for discrete values.
    - For example, if a gym membership cost $10.00 plus $6.00 for each class, this can be represented as $y = 10 + 6c$. When represented on the coordinate plane, the relationship is graphed using the points $(0,10)$, $(1,16)$, $(2,22)$, and so on.
- For mastery of this benchmark, students should be given flexibility to represent real-world contexts with discrete values as a line or as a set of points.
- Instruction directs students to graph or interpret a representation of a context that necessitates a constraint. Discuss the meaning of multiple points on the line and their meanings in the associated context (*MTR.4.1*). Allow students to discover that some points do not make sense in context and therefore should not be included in a formal solution (*MTR.6.1*). Ask students to determine which parts of the line create sensible solutions and guide them to make constraints to represent these sections.
- Instruction includes the use of technology to develop the understanding of constraints.

Common Misconceptions or Errors

- Students may assign their constraints to the incorrect variable.
- Students may miss the need for compound inequalities in their constraints. Students may not include zero as part of the domain or range.
    - For example, if a constraint for the domain is between 0 and 10, a student may forget to include 0 in some contexts, since they may assume that one cannot have negative people.
- Students may misunderstand representing interval notation from smallest to largest when representing the domain or range. To address this misconception, have students sketch the graph to demonstrate their understanding of representing the domain from left to right and range from bottom to top of their graph.

Instructional Tasks

Instructional Task 1(MTR.7.1)

Karina and Mickey are starting a catering business. They decide to rent a commercial kitchen which has a $500 security deposit and costs $30 per hour.

Part A. Identify the independent and dependent variables.
Part B. Graph the relationship.
Part C. Are there any constraints? If so, describe them.
Part D. On average for every catering job, they spend 2 hours in the kitchen prepping and charge $150 to clients for the prep time. Determine how many hours it will take to break even.
Part E. If Mickey and Karina book and fulfill 6 jobs in one month, how much profit will they expect to make?
Part F. How many catering jobs will Karina and Mickey need to fulfill in order to reach a net profit of $400?

Instructional Items

Instructional Item 1

The temperature for today, in degrees Fahrenheit (°F), can be modeled by the equation $T(x) = 64 + 1.5x$, where $T(x)$ represents the temperature x hours after 6 a.m. The high temperature for the day is expected to be 75°F. Graph the function in terms of the context.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.AR.5 *Write, solve and graph exponential and logarithmic equations and functions in one and two variables.*

MA.912.AR.5.3

Benchmark

MA.912.AR.5.3 Given a mathematical or real-world context, classify an exponential function as representing growth or decay.

Benchmark Clarifications:
Clarification 1: Within the Algebra I course, exponential functions are limited to the forms $(x) = ab^x$, where b is a whole number greater than 1 or a unit fraction, or $f(x) = a(1 \pm r)^x$, where $0 < r < 1$.

Connecting Benchmarks/Horizontal Alignment

- MA.912.F.1.6, MA.912.F.1.8
- MA.912.FL.3.1, MA.912.FL.3.2, MA.912.FL.3.4
- MA.912.DP.2.9

Terms from the K-12 Glossary

- Exponential function

Vertical Alignment

Previous Benchmarks

- MA.912.AR.1.1
- MA.912.AR.5.3

Next Benchmarks

- MA.912.AR.5.7
- MA.912.F.3.2, MA.912.F.3.7

Purpose and Instructional Strategies

In Algebra I, students identified and described exponential functions in terms of growth or decay rates. In Math for College Liberal Arts, students strengthen their understanding of exponential functions and how they are characterized by having a constant percent of change per unit interval.

- Instruction includes the connection to growth or decay of a function as a key feature (constant percent rate of change) of an exponential function and should make connections to MA.912.AR.5.5 where students will determine and interpret this key feature.
- Instruction includes the use of graphing technology to explore exponential functions.
  - For example, students can explore the function $f(x) = ab^x$ and how the a-value and b-value are affected. Ask questions such as "What impact does changing the value of a have on the graph? What about b? What values for b cause the function to increase? What values cause it to decrease?" As students explore, formalize the terms exponential growth and decay when appropriate.
  - Expand this exploration to include transformations of the function $f(x) = ab^x$. With each transformation, discuss whether or not the function has changed from growth to decay or decay to growth.
- As students solidify their understanding of $f(x) = ab^x$, use graphing technology again to have them explore the form $f(x) = a(1 \pm r)^x$. Guide students to use the sliders for a and r to visualize that only r determines whether the function represents exponential growth or exponential decay.

- Have students discuss which values for r cause exponential growth or decay. They should observe that negative values cause exponential decay while positive values cause exponential growth.
- Instruction includes a focus on real-world context when approaching problems within this benchmark. Instruction includes discussion on the natural asymptotes that are apparent in real-world contexts (*MTR.4.1*). Experiences can include but are not limited to population, epidemics, value problems and interest-bearing bank accounts. This benchmark is a foundation for students' work within MA.912.FL.3.1, MA912.FL.3.2 and MA.912.FL.3.4.
- Instruction includes students using an irrational number, symbolized by the letter e, as the base in many applied exponential functions. The number e is called the natural base and can be approximated to 2.72.

Common Misconceptions or Errors

- Students may not understand that growth factors have one constraint $(b > 1)$ while decay factors have a compound constraint $(0 < b < 1)$. Some students may think that as long as $b < 1$, the function will represent exponential decay.
- Students may think that if a is negative and $r > 0$ or $b > 1$, the function represents an exponential decay. To address this misconception, help students understand that the negative values are growing at an exponential rate.

Instructional Tasks

Instructional Task 1 (MTR.2.1)

Edison buys a mid-range capacity tractor for $32,500. The value of the tractor depreciates by 8.3% per year.

Part A. Create a graph or table of values to represent the given situation.
Part B. Create an exponential function that describes the given situation.
Part C. Does this situation represent an exponential growth or decay function?

Instructional Items

Instructional Item 1

Given the function $f(x) = 3\left(\frac{5}{2}\right)^{x-4} + 1$, does it represent an exponential growth or decay function?

Instructional Item 2

Every 124 minutes, $\frac{1}{2}$ of a drug dosage remains in the body. Does this situation represent an exponential growth or decay function?

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.AR.5.4

Benchmark

MA.912.AR.5.4 Write an exponential function to represent a relationship between two quantities from a graph, a written description or a table of values within a mathematical or real-world context.

Benchmark Clarifications:
Clarification 1: Within the Algebra I course, exponential functions are limited to the forms $f(x) = ab^x$, where b is a whole number greater than 1 or a unit fraction, or $f(x) = a(1 \pm r)^x$, where $0 < r < 1$.
Clarification 2: Within the Algebra I course, tables are limited to having successive nonnegative integer inputs so that the function may be determined by finding ratios between successive outputs.

Connecting Benchmarks/Horizontal Alignment

- MA.912.F.1.8
- MA.912.FL.3.1, MA.912.FL.3.2, MA.912.FL.3.4
- MA.912.DP.2.9

Terms from the K-12 Glossary

- Exponential Function

Vertical Alignment

Previous Benchmarks

- MA.912.AR.5.3, MA.912.AR.5.4
- MA.912.F.1.1

Next Benchmarks

- MA.912.AR.5.7

Purpose and Instructional Strategies

In Algebra I, students wrote exponential functions that modeled relationships characterized by having a constant percent of change per unit interval. In Math for College Liberal Arts, students continue this work, not limited to the forms $f(x) = ab^x$, where b is a whole number greater than 1 or a unit fraction, or $f(x) = a(1 \pm r)^x$, where $0 < r < 1$. In other courses, students will continue to solve problems that are modeled with exponential functions.

- Instruction includes connections with MA.912.AR.5.5 as the constant percent rate of change per unit interval is a critical component of an exponential function.
- Instruction includes guidance on how to determine the initial value or the percent rate of change of an exponential function when it is not provided.
    - For example, if the initial value of $(0,3)$ is given, students can now write the function as $f(x) = 3b^x$. Guide students to choose a point on the curve that has integer coordinates such as $(2,12)$. Lead them to substitute the point into their function to find b. Students should recognize that exponential functions are restricted to positive values of b, leading to the function $f(x) = 3(2)^x$.
- Problems include cases where the initial value is not given.
- Use prior knowledge on transformations of functions to explore different forms of exponential functions.
    - For example, the function $f(x) = 16\left(\frac{1}{2}\right)^x$ can be written as $f(x) = 4\left(\frac{1}{2}\right)^{x-2}$ from the given table of values, showing the horizontal shift.

x	0	1	2	3
$f(x)$	16	8	4	2

- Instruction includes students using an irrational number, symbolized by the letter e, as the base in many applied exponential functions. The number e is called the natural base and can be approximated to 2.72. It is used when the relationship between two quantities is growing exponentially at a continuous rate.
  - Include real-world problems on compound interest and continuously compounded interest in connection with MA.912.FL.3.4 to highlight the use of e.

Common Misconceptions or Errors

- Students may not understand when to use an exponential function based on a table of values or written description.

Instructional Tasks

Instructional Task 1 (MTR.7.1)

Across the state of Florida, 128 basketball teams are chosen to participate in a championship tournament. After each round, half the teams are eliminated.

Part A. Write a function to represent the number of teams remaining in the tournament after x number of rounds played.

Part B. Describe the number of teams remaining in terms of the rate of change.

Part C. Graph the function and describe its key features.

Part D. Are there any constraints on the domain or range?

Instructional Items

Instructional Item 1

A class science experiment to measure the growth of bacteria starts with 70 cells that triple every hour. Write a function to model this situation.

Instructional Item 2

Ian was 60 inches tall in January and is growing every month. His height is recorded in the table below. Create a function to represent Ian's growth after m months.

Month	Height (inches)
January	60
February	60.6
March	60.206
April	61.818

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.AR.5.5

Benchmark

MA.912.AR.5.5	Given an expression or equation representing an exponential function, reveal the constant percent rate of change per unit interval using the properties of exponents. Interpret the constant percent rate of change in terms of a real-world context.

Connecting Benchmarks/Horizontal Alignment

- MA.912.F.1.6, MA.912.F.1.8
- MA.912.FL.3.4
- MA.912.DP.2.9

Terms from the K-12 Glossary

- Exponential function

Vertical Alignment

Previous Benchmarks

- MA.912.NSO.1.2
- MA.912.AR.5.3, MA.912.AR.5.4
- MA.912.F.1.8

Next Benchmarks

- MA.912.AR.5.7, MA.912.AR.5.8, MA.912.AR.5.9

Purpose and Instructional Strategies

In Algebra I, students generated equivalent algebraic expressions suing the properties of exponents and wrote exponential functions that modeled relationships characterized by having a constant percent of change per unit interval. In Math for College Liberal Arts, students rewrite these exponential functions to reveal the constant percent rate of change per unit interval and interpret its meaning in terms of the given context. In other courses, students will solve and graph mathematical and real-world problems that are modeled with exponential functions, including interpreting key features in terms of the context.

- Students may need support with using the properties of exponents fluently.
- Instruction includes connecting prior knowledge of constant rate of change, including linear functions and making comparisons as noted in MA.912.F.1.6.
- When generating equivalent expressions to reveal the constant percent rate of change per unit interval, students should be encouraged to approach from different entry points and discuss how they are different but equivalent strategies (*MTR.2.1*).
- Instruction includes interpreting percentages of growth/decay from exponential functions and see that b can be used to determine a percentage.
  - For example, the function $f(x) = 500(1.16)^x$ represents 16% growth of an initial value.
    - Guide students to discuss the meaning of the number 1.16 as a percent. They should understand it represents 116%. Taking 116% of an initial value increases the magnitude of that value. (Students can test this in a calculator to confirm.) Taking this percentage repetitively leads to exponential growth.
  - For example, the function $f(x) = 500(0.72)^x$ represents 28% decay of an initial value.
    - Guide students to discuss the meaning of the number 0.72 as a percent.

They should understand it represents 72%. Taking 72% of an initial value decreases the magnitude of that value. (Students can test this in a calculator to confirm.) Taking this percentage repetitively leads to exponential decay.

- For example, the function $f(x) = 500(1)^x$ represents an initial value that neither grows nor decays as x increases.
  - Guide students to discuss the meaning of the number 1 when it comes to growth/decay factors. They should understand it represents 100%. Taking 100% of an initial value causes the value to remain the same. (Students can test this in a calculator to confirm.) Taking this percentage repetitively leads to no change in the initial value (explaining the horizontal line that shows when b=1on the graph).
- Instruction includes real-world problems on compound interest and continuously compounded interest in connection with MA.912.FL.3.4.

Common Misconceptions or Errors

- Students may not have fully mastered the Laws of Exponents and understand the mathematical connections between the bases and the exponents.
- Students may struggle with representing the growth factor as a percent, rather than a whole number.
- Students may represent the growth rate or decay rate within the formula instead of using the formula $f(x) = a(1 \pm r)^x$. To address this misconception, students should record the values being used to showcase the growth or decay to see their mistake.

Instructional Tasks

Instructional Task 1 (MTR.2.1, MTR.5.1)

Four physicists describe the amount of a radioactive substance in grams, Q, left after t years below.

Function 1: $Q = 300e^{-0.0577t}$

Function 2: $Q = 300\left(\frac{1}{2}\right)^{\frac{t}{12}}$

Function 3: $Q = 300(0.9439)^t$

Function 4: $Q = 252.290(0.9439)^{t-3}$

Part A. Compare the four different functions describing the radioactive substance.
Part B. Determine whether the described functions are equivalent.
Part C. What is the constant percent rate of change annually of the radioactive substance?
Part D. Why do you think each of the four physicists described the amount of radioactive substance differently? What information does each tell you?

Instructional Task 2 (MTR.2.1, MTR.5.1)

A fisherman illegally introduces some fish into a lake, and they quickly propagate. The growth of the population of this new species (within a period of a few years) is modeled by $P(x) = 5b^x$, where x is the time in weeks following the introduction and b is a positive unknown base.

Part A. Exactly how many fish did the fisherman release into the lake?

Part B. Find b if you know the lake contains 33 fish after eight weeks.
Part C. Instead, now suppose that $P(x) = 5b^x$ and $b = 2$. What is the weekly percent growth rate in this case? What does this mean in everyday language?
Part D. How does the weekly percent growth rate compare in Part B and Part C?

Instructional Items

Instructional Item 1

The equation $y = 42{,}500(0.9198)^x$ represents the value of a car x years after its initial purchase. The average rate of depreciation for vehicles is often measured in 5-year intervals. Write an equivalent expression to show the constant percent rate of change over 5 years.

Instructional Item 2

The function $V(t) = 250{,}000(1.038)^t$ represents the value (V) of a home t years after its initial purchase. Interpret the annual percent rate of change in this context.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.AR.5.6

Benchmark

MA.912.AR.5.6 Given a table, equation or written description of an exponential function, graph that function and determine its key features.

Benchmark Clarifications:
Clarification 1: Key features are limited to domain; range; intercepts; intervals where the function is increasing, decreasing, positive or negative; constant percent rate of change; end behavior and asymptotes.
Clarification 2: Instruction includes representing the domain and range with inequality notation, interval notation or set-builder notation.
Clarification 3: Within the Algebra I course, notations for domain and range are limited to inequality and set-builder.
Clarification 4: Within the Algebra I course, exponential functions are limited to the forms $f(x) = ab^x$, where b is a whole number greater than 1 or a unit fraction, or $f(x) = a(1 \pm r)^x$, where $0 < r < 1$.

Connecting Benchmarks/Horizontal Alignment

- MA.912.F.1.6, MA.912.F.1.8
- MA.912.DP.2.9

Terms from the K-12 Glossary

- Coordinate plane
- Domain
- Exponential Function
- Function Notation
- Range
- x-intercept
- y-intercept

Vertical Alignment

Previous Benchmarks

- MA.912.F.1.5, MA.912.F.1.6

Next Benchmarks

- MA.912.AR.5.7, MA.912.AR.5.8, MA.912.AR.5.9
- MA.912.AR.9.10
- MA.912.F.3.7

Purpose and Instructional Strategies

In Algebra I, students graphed exponential functions and determined their key features, including asymptotes and end behavior. In Math for College Liberal Arts, students continue this work, without the limitations on function forms given in Algebra I. In other courses, students will solve and graph mathematical and real-world problems that are modeled with exponential functions, interpreting key features of exponential functions in terms of the context, and working with logarithms, showing they are the inverse of exponentials.

- For students to have full understanding of exponential functions, instruction includes MA.912.AR.5.3 and MA.912.AR.5.4. Growth or decay of a function can be defined as a key feature (constant percent rate of change) of an exponential function and useful in understanding the relationships between two quantities.
- Students are first introduced to asymptotes in Algebra I. Asymptotes are important in the study of other types of functions, including rational functions, which are included in other high school courses.
- Instruction provides the opportunity for students to explore the meaning of an asymptote graphically and algebraically. Through work in this benchmark, students will discover asymptotes are useful guides to complete the graph of a function, especially when drawing them by hand. For mastery of this benchmark, asymptotes can be drawn on the graph as a dotted line or not drawn on the graph.
    - Instruction should not be limited to exponential functions of the form $f(x) = ab^x$. Students should explore how asymptotes move as this parent function is translated.
- Emphasize reading a graph from left to right when identifying key features such as increasing and decreasing intervals.
- When using real-world problems, discuss the meaning of key features in context. Build on students' prior knowledge of interpreting intercepts and rate of change.
- Instruction includes the use of x-y notation and function notation.
- Instruction includes representing domain, range and intervals where the function is increasing, decreasing, positive or negative, using words, inequality notation, set-builder notation and interval notation.
    - Words
      If the domain is all real numbers, it can be written as "all real numbers" or "any value of x, such that x is a real number."
    - Inequality notation
      If the domain is all values of x greater than 2, it can be represented as $x > 2$.
    - Set-builder notation
      If the range is all values of y less than or equal to zero, it can be represented as $\{y|y \leq 0\}$ and is read as "all values of y such that y is less than or equal to zero."
    - Interval notation
      If the domain is all values of x less than or equal to 3, it can be represented as

$(-\infty, 3]$. If the domain is all values of x greater than 3, it can be represented as $(3, \infty)$. If the range is all values greater than or equal to -1 but less than 5, it can be represented as $[-1, 5)$.

Common Misconceptions or Errors

- Students may mistake interval notation with open ends as a coordinate point.
- Students may confuse when an exponential function is needed, rather than a linear or quadratic function, given a table or written description.
- Students may struggle with identifying asymptotes, especially when they are not represented by an axis.

Instructional Tasks

Instructional Task 1 (MTR.3.1, MTR.4.1)

Using the function $f(x) = 50(0.8)^x$ determine the following:

Part A. Graph the function.
Part B. What are the domain and range of this function?
Part C. What are the end behaviors of this function?
Part D. Is the function increasing or decreasing? Explain.
Part E. What is the constant percent rate of change for this function?
Part F. Are there any asymptotes for this function? If so, describe them.

Instructional Task 2 (MTR.3.1, MTR.4.1)

The initial value of an exponential function is 300 and has a growth factor of 13%.

Part A. Graph the function.
Part B. What are the domain and range of this function?
Part C. What are the end behaviors of this function?
Part D. Is the function increasing or decreasing? Explain.
Part E. What is the constant percent rate of change for this function?
Part F. Are there any asymptotes for this function? If so, describe them.

Instructional Items

Instructional Item 1

Given the table of values below, graph the function and describe its key features.

x	y
1	-4
2	-1
3	$-\frac{1}{4}$

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

Functions

MA.912.F.1 *Understand, compare and analyze properties of functions.*

MA.912.F.1.6

Benchmark

MA.912.F.1.6 Compare key features of linear and nonlinear functions each represented algebraically, graphically, in tables or written descriptions.

Benchmark Clarifications:
Clarification 1: Key features are limited to domain; range; intercepts; intervals where the function is increasing, decreasing, positive or negative; end behavior and asymptotes.
Clarification 2: Within the Algebra I course, functions other than linear, quadratic or exponential must be represented graphically.
Clarification 3: Within the Algebra I course, instruction includes verifying that a quantity increasing exponentially eventually exceeds a quantity increasing linearly or quadratically.

Connecting Benchmarks/Horizontal Alignment

- MA.912.AR.2.5
- MA.912.AR.5.5, MA.912.AR.5.6
- MA.912.FL.3.1, MA.912.FL.3.4

Terms from the K-12 Glossary

- Domain
- Intercept
- Range
- Slope

Vertical Alignment

Previous Benchmarks

- MA.912.AR.2.4
- MA.912.AR.3.7
- MA.912.AR.4.3
- MA.912.AR.5.6
- MA.912.F.1.5
- MA.912.GR.7.3

Next Benchmarks

Purpose and Instructional Strategies

In Algebra I, students compared key features of two or more linear or nonlinear functions. In Math for College Liberal Arts, students compare key features of linear and exponential functions represented graphically, algebraically, or with written descriptions. In other courses, students will continue to compare key features based on the function types that are the main theme of the course.

- In this benchmark, instruction includes comparing a linear and an exponential function or two exponential functions. The number of functions being compared is not limited to two.
- Problem types include comparing linear to exponential functions that are represented in different forms.
- Instruction includes student exploration of linear and exponential models to ultimately determine that a quantity increasing exponentially eventually exceeds a quantity increasing linearly.
  - For example, provide the following context:

- You are being contracted by a large company to provide technical services to a major engineering project. The contract will involve you advising a group of engineers for three weeks. The company offers you a choice of two methods of payment for your services. The first is to receive $500 per day of work. The second is to receive payment on a scale: two cents for the first day of work, four cents for the second day, eight cents for the third day, etc. Which method would you choose?
  Have students choose a method of payment and begin a class discussion regarding the reasoning students used to make their choices (*MTR.4.1*). Ask if students can create a function to represent each payment method (*MTR.7.1*).

- Instruction in Algebra I included representing domain, range and constraints using words, inequality notation and set-builder notation. In Math for College Liberal Arts, instruction will also include interval notation.
  - Words
    If the domain is all real numbers, it can be written as "all real numbers" or "any value of x, such that x is a real number."
  - Inequality notation
    If the domain is all values of x greater than 2, it can be represented as $x > 2$.
  - Set-builder notation
    If the range is all values of y less than or equal to zero, it can be represented as $\{y|y \leq 0\}$ and is read as "all values of y such that y is less than or equal to zero."
  - Interval notation
    If the domain is all values of x less than or equal to 3, it can be represented as $(-\infty, 3]$. If the domain is all values of x greater than 3, it can be represented as $(3, \infty)$. If the range is all values greater than or equal to -1 but less than 5, it can be represented as $[-1, 5)$.
- Instruction includes discussions of key features in connection with MA.912.AR.2.5 and MA.912.AR.5.6.

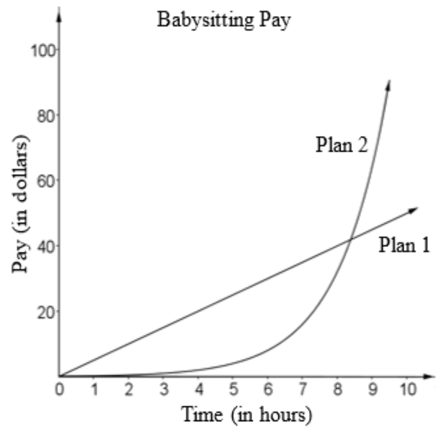
Common Misconceptions or Errors

- When describing key features, students may assign their constraints to the incorrect variable. In these cases, ask reflective questions to help students examine the meaning of the key features in the problem.
- Students may also miss the need for compound inequalities when describing domain or range. In these cases, use a graph of the function to point out areas of their constraint that would not make sense in context.

Instructional Tasks

Instructional Task 1 (MTR.4.1, MTR.5.1)

Sara has been asked to babysit for a neighbor. She is offered two payment options. With the first plan, she is paid $5.00 per hour. With the second plan, she is paid $0.25 for one hour, $0.50 for two hours, $1.00 for three hours, and so on, as shown in both the graph and the table.



Hours	Plan 1	Plan 2
1	5.00	0.25
2	10.00	0.50
3	15.00	1.00
4	20.00	2.00
5	25.00	4.00
6	30.00	8.00
7	35.00	16.00
8	40.00	32.00
9	45.00	64.00
10	50.00	128.00

Part A. What type of function is being represented in Plan 1 and Plan 2? How do you know?
Part B. What are the domain and range of Plan 1 and Plan 2?
Part C. What key features in Plan 1 and Plan 2 are the same?
Part D. In terms of context, what does the intersection of Plan 1 and 2 represent?
Part E. When is it appropriate to choose Plan 1? Why?
Part F. When is it appropriate to choose Plan 2? Why?

Instructional Task 2 (MTR.4.1)

The Plasterer family bought two plots of land. One plot of land was purchased for $50,000 in the A section of the city. The second plot of land was purchased for $40,000 in the B section of the city. The value of the land in the A section of the city can be represented by the function $f(x) = 50{,}000(1.075)^x$. The value of the land in the B section of the city is represented in the table below.

x	0	1	2	3
$g(x)$	40,000	44,000	48,400	64,400

Part A. What are the key features for the plot of land in the A section of the city?
Part B. What are the key features for the plot of land in the B section of the city?
Part C. Is the end behavior for the two plots of land the same or different? How do you know?
Part D. Which plot's value is increasing more quickly? How do you know?
Part E. Which plot of land would it make sense to sell first? How did you determine your decision?

Instructional Items

Instructional Item 1

The functions $f(x)$ and $g(x)$ are shown below.

$$f(x) = 300(0.95)^x$$

x	2	4	5	7
$g(x)$	256	164	131	84

The function $g(x)$ has a decay factor of 0.8. Which function has a greater y-intercept?

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.F.1.8

Benchmark

MA.912.F.1.8 Determine whether a linear, quadratic or exponential function best models a given real-world situation.

Benchmark Clarifications:
Clarification 1: Instruction includes recognizing that linear functions model situations in which a quantity changes by a constant amount per unit interval; that quadratic functions model situations in which a quantity increases to a maximum, then begins to decrease or a quantity decreases to a minimum, then begins to increase; and that exponential functions model situations in which a quantity grows or decays by a constant percent per unit interval.
Clarification 2: Within this benchmark, the expectation is to identify the type of function from a written description or table.

Connecting Benchmarks/Horizontal Alignment

- MA.912.F.1.6
- MA.912.DP.2.4, MA.912.DP.2.9

Terms from the K-12 Glossary

- Exponential function
- Linear function
- Quadratic function

Vertical Alignment

Previous Benchmarks

- MA.912.AR.2.5
- MA.912.AR.3.8
- MA.912.DP.2.4

Next Benchmarks

- MA.912.AR.5.7
- MA.912.AR.9.7, MA.912.AR.9.10
- MA.912.DP.2.8

Purpose and Instructional Strategies

In Algebra I, students determined whether a linear, quadratic or exponential function best modeled a situation. In Math for College Liberal Arts, students will continue this work focusing on linear and exponential functions. In future courses, students will fit linear, quadratic and exponential functions to statistical data.

- Instruction includes opportunities for students to work with context that involves financial literacy and exponential decay/growth situations. While quadratic relationships are not part of the expectation for Math for College Liberal Arts, students do have knowledge of quadratics from Algebra I.
- Instruction includes identifying function types from tables and from written descriptions.
  - When examining written descriptions, guide students to see that linear functions model situations in which a quantity changes by a constant amount per unit interval.
  - When examining written descriptions, instruction includes exponential functions modeling situations in which a quantity grows or decays by a constant percent per unit interval.
- When considering tables, instruction guides students to understand that linear relationships have a common difference per unit interval (or a constant rate of change), quadratic relationships produce a common second difference (as demonstrated in Algebra I), and exponential relationships produce a constant percent of change per unit interval

(*MTR.5.1*).

- Considering tables like the one below, lead students to discover that there is a common difference of 0.4 between successive y-values. Plotting these points using graphing software will verify that they are collinear.

x	-4	-3	-2	-1	0
y	1.6	2.0	2.4	2.8	3.2
1st Difference	0.4 0.4 0.4 0.4				

- Considering tables like the one below, as demonstrated in Algebra I, were lead to discover that there is a common second difference of -8 between successive y-values. Plotting these points using graphing software will verify that they form a parabola.

x	-2	-1	0	1	2
y	-14	-2	2	-2	-14
1st Difference	12 4 -4 12				
2nd Difference	-8 -8 -8				

- Considering tables like the one below, lead students to discover that there is no common difference or second difference. In this case, there is a common ratio of $\frac{3}{1}$ between successive y-values. Plotting these points using graphing software will verify that they form an exponential graph.

x	2	4	6	8	10
y	3	9	27	81	243
1st Difference	6 18 54 162				
2nd Difference	12 36 108				
Common Ratio	$\frac{3}{1}$ $\frac{3}{1}$ $\frac{3}{1}$ $\frac{3}{1}$				

- Students should note that the search for common differences and ratios only works when the x-values are equidistant from each other. Lead them to check for this when presented with tables of values to consider.

Common Misconceptions or Errors

- Students may interpret any relationship that increases/decreases at a non-constant rate as being an exponential relationship. Have these students verify an exponential relationship by looking for common ratios. If they are interpreting a written description, direct them to make a sample table of values from the context to examine.
- Some students may miscalculate first and second differences that deal with negative values, especially if they perform them mentally. In these cases, have students quickly write out the subtraction expression [i.e., $-14-(-2)$].

Instructional Tasks

Instructional Task 1 (MTR.4.1, MTR.7.1)

A car was purchased for the total price (including taxes and fees) of $33,124.27. The value of the car decreases by 15% each year.

Part A. What type of function could be used to represent this situation?

Part B. Justify your reasoning.

Instructional Task 2 (MTR.4.1, MTR.7.1)

Every March, there is a college basketball tournament that invites 64 teams to play. Each round half of the teams are eliminated from the competition. How many teams are remaining after 3 rounds?

Part A. What type of function could be used to represent this situation?

Part B. Justify your reasoning.

Part C. How many teams are remaining after 3 rounds?

Instructional Items

Instructional Item 1

The table below shows the profit p, in thousands of dollars, of a company during its first six years.

x	1	2	3	4	5	6
$p(x)$	24	29	34	39	44	49

Which type of function best models the data?

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.FL.3 *Describe the advantages and disadvantages of short-term and long-term purchases.*

MA.912.FL.3.1

Benchmark

MA.912.FL.3.1 Compare simple, compound and continuously compounded interest over time.

Benchmark Clarifications:
Clarification 1: Instruction includes taking into consideration the annual percentage rate (APR) when comparing simple and compound interest.

Connecting Benchmarks/Horizontal Alignment

- MA.912.F.1.6, MA.912.F.1.8
- MA.912.AR.2.5
- MA.912.AR.5.3, MA.912.AR.5.4, MA.912.AR.5.5

Terms from the K-12 Glossary

- Exponential function
- Simple interest

Vertical Alignment

Previous Benchmarks

- MA.912.FL.3.2

Next Benchmarks

- MA.912.FL.4.3, MA.912.FL.4.4

Purpose and Instructional Strategies

In Algebra I, students solved problems involving simple and compound interest, using arithmetic operations and graphing. In Math for College Liberal Arts, students will compare simple, compound and continuously compounded interest over time in a variety of real-world contexts. In later courses, students will describe the advantages and disadvantages of financial and investment plans.

- In this benchmark, students will describe the difference between simple and compound interest. Instruction should feature a variety of real-world contexts related to money and business.
    - The simple interest formula, $I = prt$, calculates only the interest earned over time. Each year's interest is calculated from the initial principal, not the total value of the investment of that point in time. Lead students to generate the formula by exploring patterns *(MTR.5.1)*.
      Considering the following:
        - Jalen invests $100 in a bank account that pays 5% simple interest each year. How much interest does Jalen earn in 15 years? What is the total value of his investment in 15 years?
        - Have students begin exploration by creating a table like the one below.

Year	Interest Earned	Total Amount
1	$100(0.05)$	$100(0.05) + 100$
2	$100(0.05) \times 2$	$100(0.05) \times 2 + 100$
3	$100(0.05) \times 3$	$100(0.05) \times 3 + 100$

4	$100(0.05) \times 4$	$100(0.05) \times 4 + 100$
t	$p(r) \times t$ or prt	$p(r) \times t + p$ or $prt + p$ or $p(1 + rt)$

- Be sure to highlight the difference between the simple interest formula and those for compound interest. The formula $I = prt$ only calculates the interest, whereas the compound interest formulas calculate the worth of the total investment.

- The simple interest amount formula, $A = P(1 + rt)$, calculates the total value of an investment over time.
- The compound interest formula, $A = P\left(1 + \frac{r}{n}\right)^{nt}$, also calculates the *total value* of an investment over time. Each month/year's interest is calculated from the total value of the investment of that point in time. Lead students to generate the formula by exploring patterns *(MTR.5.1)*.
  Consider the following:
  - Jalen invests $100 in a bank account that pays 5% interest compounded quarterly. How much interest does Jalen earn in 15 years? What is the total value of his investment in 15 years?
  - Guide students through an exploration beginning with the simple interest formula.
    - The interest for one quarter would be
      $$I = prt$$
      $$I = (100)(0.05)\left(\frac{1}{4}\right)$$
      $$I = 100\left(\frac{0.05}{4}\right)$$
    - The total amount of the investment after one quarter would then be
      $$A_1 = 100 + 100\left(\frac{0.05}{4}\right)$$
      $$A_1 = 100\left(1 + \frac{0.05}{4}\right)$$
    - The value of the investment after 2 quarters, A_2, would be
      $$A_2 = A_1 + A_1\left(\frac{0.05}{4}\right)$$
      $$A_2 = A_1\left(1 + \frac{0.05}{4}\right)$$
      $$A_2 = 100\left(1 + \frac{0.05}{4}\right)\left(1 + \frac{0.05}{4}\right)$$
      $$A_2 = 100\left(1 + \frac{0.05}{4}\right)^2$$
    - The value of the investment after 3 quarters, A_3, would be
      $$A_3 = A_2 + A_2\left(\frac{0.05}{4}\right)$$
      $$A_3 = A_2\left(1 + \frac{0.05}{4}\right)$$

$$A_3 = 100\left(1+\frac{0.05}{4}\right)\left(1+\frac{0.05}{4}\right)\left(1+\frac{0.05}{4}\right)$$

$$A_3 = 100\left(1+\frac{0.05}{4}\right)^3$$

- The value of the investment after 4 quarters, A_4, would be

$$A_4 = A_3 + A_3\left(\frac{0.05}{4}\right)$$

$$A_4 = A_3\left(1+\frac{0.05}{4}\right)$$

$$A_4 = 100\left(1+\frac{0.05}{4}\right)\left(1+\frac{0.05}{4}\right)\left(1+\frac{0.05}{4}\right)\left(1+\frac{0.05}{4}\right)$$

$$A_4 = 100\left(1+\frac{0.05}{4}\right)^4$$

- At this point, student should see the connection between the exponent 4 and the number of quarters the investment has accumulated interest. Speed the exploration up by moving to 8, 12 and 16 quarters to build the idea of the exponent being the product of the number of years and the number of times the investment compounds each year.
  - After 8 quarters (2 years), A_8, would be

$$A_8 = 100\left(1+\frac{0.05}{4}\right)^8$$

$$A_8 = 100\left(1+\frac{0.05}{4}\right)^{4(2)}$$

  - After 12 quarters (3 years), A_{12}, would be

$$A_{12} = 100\left(1+\frac{0.05}{4}\right)^{12}$$

$$A_{12} = 100\left(1+\frac{0.05}{4}\right)^{4(3)}$$

- Students should now see the pattern emerge for the compound interest formula. Have them replace the numbers in the equation to generate $A = P\left(1 + \frac{r}{n}\right)^{nt}$.

- Ask students how they would describe the difference between the simple and compound interested to someone who does not know (*MTR.4.1*).
- Instruction guides students to consider the concept of continuous compounding by examining investment situations where compounding is done increasingly more often (*MTR.5.1*).
  - The continuously compounded interest formula, $A = Pe^{rt}$, calculates the total value of an investment over time. Instead of calculating interest on a finite number of compounding periods (e.g., monthly, annually, daily), continuous compounding calculates interest assuming a constant compounding over an infinite number of compounding periods. Continuously compounded interest assumes interest is compounded and added to the amount accrued an infinite number of times.

- Instruction includes defining continuous compounding. This includes, but is not limited to daily, weekly, monthly, quarterly, semiannually and annually.
- Instruction includes comparison of simple, compound and continuously compounded interest using various representations, such as graphs, tables, equations and written description. Students will be guided to interpret key features of linear and exponential relationships in terms of context to connect to simple and compound interest problems.
- Instruction includes the Annual Percentage Yield (AYP), $Y = \left(1 + \frac{r}{n}\right)^n - 1$. The Annual Percentage Yield allows you to compare compound interest rates with differing compounding periods.
    - For example, a student may be asked to determine which a better investment is: 1.28% compounded weekly or 1.35% compounded monthly.

Compounded Weekly	Compounded Monthly
$Y = \left(1 + \frac{0.0128}{52}\right)^{52} - 1$	$Y = \left(1 + \frac{0.0135}{12}\right)^{12} - 1$
$Y = (1.000246)^{52} - 1$	$Y = (1.001125)^{12} - 1$
$Y = 1.012881 - 1$	$Y = 1.013584 - 1$
$Y = 0.012881$	$Y = 0.013584$
$Y = 1.2881\%$	$Y = 1.3584\%$

    The better investment would be compounded monthly at 1.35%
- Instruction includes the discussion of Annual Percentage Rate (APR) – the total interest that will be paid in the year. If the interest is paid in smaller time increments, the APR will be divided up.

Common Misconceptions or Errors

- Many institutions advertise investments by using the annual percentage yield (APY) rather than the annual percentage rate (APR) since it is greater for investments that compound multiple times annually. Students should be careful when analyzing investments to see which percentage is being advertised.
- For simple interest scenarios, students may forget to add the principal to the interest they calculate using the simple interest formula when calculating the total amount of an investment over time.
- For compound interest scenarios, students may forget to subtract the principal from the total amount they calculate using a compound interest formula when calculating the total interest earned over time.
- Students may confuse the compounding period variable with time variable.
- Students may assume continuously compounding accounts will have a substantially larger returns, when in fact, the difference in the total interest earned through continuous compounding is not very high when compared to traditional compounding periods.
- Students may incorrectly use the rate.
    - For example, students may be given the rate of 4.2% and they use 4.2 instead of 0.042. To address this misconception, students should use multiple representations of percentages.

Instructional Tasks

Instructional Task 1 (MTR.5.1)

A principal investment of $5,000 had an interest rate of 3%.

Part A. Complete the table below.

Years Invested	Simple Interest	Compound Interest (yearly)	Compound Interest (monthly)	Compound Interest (quarterly)	Compound Interest (Continuous)
1					
2					
3					
4					
5					

Part B. After 5 years, what is the best way to invest the money?

Part C. Which investment will have the greatest value in 15 years? Explain why.

Part D. Which investment will have the least value in 15 years? Explain why.

Instructional Items

Instructional Item 1

To the nearest cent, how much more does an investment of $3500 earn over 10 years, compounded continuously at 3.5%, than a $3500 investment over 10 years, compounded quarterly at 3.5%?

Instructional Item 2

Which is the better financial investment: 3.85% compounded weekly or 3.74% compounded monthly?

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.FL.3.2

Benchmark

MA.912.FL.3.2 Solve real-world problems involving simple, compound and continuously compounded interest.

Example: Find the amount of money on deposit at the end of 5 years if you started with $500 and it was compounded quarterly at 6% interest per year.

Example: Joe won $25,000 on a lottery scratch-off ticket. How many years will it take at 6% interest compounded yearly for his money to double?

Benchmark Clarifications:

Clarification 1: Within the Algebra I course, interest is limited to simple and compound.

Connecting Benchmarks/Horizontal Alignment

- MA.912.AR.2.5
- MA.912.AR.5.4, MA.912.AR.5.5
- MA.912.DP.2.4, MA.912.DP.2.9
- MA.912.F.1.8

Terms from the K-12 Glossary

- Simple interest

Vertical Alignment

Previous Benchmarks

- MA.7.AR.3.1
- MA.912.NSO.1.1

Next Benchmarks

- MA.912.FL.3.3

Purpose and Instructional Strategies

In Algebra I, students solved problems involving simple and compound interest, using arithmetic operations and graphing. In Math for College Liberal Arts, students solve compound interest and continuously compounded interest problems. In other courses, students will describe the advantages and disadvantages of financial and investment plans and also solve compound interest problems to determine lengths of time, including those that require the use of logarithms.

- In this benchmark, students solve real-world problems that feature simple, compound, and continuously compounded interest problems in real-world setting. The expectation of this course is not to use logarithms to solve problems.
- Instruction directs students to compare real-world investment situations to highlight the differences between simple and compound interest by engaging in non-routine discussion-based tasks (*MTR.7.1, MTR1.1*).
  - For example, City Bank pays a simple interest rate of 3% per year, meaning that each year the balance increases by 3% of the initial deposit. National Bank pays an compound interest rate of 2.6% per year, compounded monthly, meaning that each month the balance increases by one twelfth of 2.6% of the previous month's balance.
    - Which bank will provide the largest balance if you plan to invest $10,000 for 10 years? For 15 years?

- Write an expression for $C(y)$, the City Bank balance, y years after a deposit is left in the account. Write an expression for $N(m)$, the National Bank balance, m months after a deposit is left in the account.
- Create a table of values indicating the balances in the two bank accounts from year 1 to year 15. For which years is City Bank a better investment, and for which years is National Bank a better investment?

- Compound interest problems presented for this benchmark may require students to generate equivalent expressions to identify and interpret certain parts of the context.
  - For example, Jason deposits $850 in an account that earns an annual interest rate of 4.8%. The interest is compounded monthly and Jason wants to determine the total amount of interest he will earn in one year. With the given information, derive that the value of the account is equal to $850\left(1+\frac{0.048}{12}\right)^{12t}$. The expression can be rewritten as $850\ [(1.004)^{12}]^t$ leading to $850(1.049)^t$ to find that the total amount of interest in a year would be approximately 4.9% of his initial investment.

Common Misconceptions or Errors

- Some problems related to this benchmark may ask students for the interest earned over a period of time while others may ask for the account balance or total value of the investment over a period of time. Some students may miss this distinction and may always calculate total interest for simple interest problems and total value for compound interest problems. In these cases, point students back to the wording of the problem and help them assess the reasonableness of their answers in context. (*MTR.6.1*)

Instructional Tasks

Instructional Task 1 (MTR.4.1, MTR.7.1)

Gibbs signs up for a new airline credit card that has a 24% annual interest rate. If he doesn't pay his monthly statements, interest on his balance would compound *daily*.

Part A. Describe what it means to have interest compound daily.

Part B. If Gibbs does not pay his credit card for 30 days, what would he have to pay the credit card company?

Part C. If Gibbs does not pay his credit card for six months, how many days is the interest compounding?

Part D. What would be the actual percentage rate he would pay the credit card company if he did not pay his credit card for six months?

Part E. If Gibbs never pays his statements for a full year, what would be the actual percentage rate he would pay the credit card company?

Instructional Task 2 (MTR.7.1)

Liz deposits $800 in a savings account that pays simple annual interest. After some months, she earns $64.80. What is the interest rate of her savings account?

Part A. What additional information is needed to solve this problem?

Part B. If Liz invests her money for 18 months, what is the interest rate for her savings account?

Part C. How would the interest rate differ if she only invested for 9 months?

Instructional Items

Instructional Item 1

Juniper deposits $525 in an account that pays 4.3% compounded continuously. If she keeps the money in the account for 12 years, how much interest will she earn?

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.FL.3.4

Benchmark

MA.912.FL.3.4	Explain the relationship between simple interest and linear growth. Explain the relationship between compound interest and exponential growth and the relationship between continuously compounded interest and exponential growth.

Benchmark Clarifications:

Clarification 1: Within the Algebra I course, exponential growth is limited to compound interest.

Connecting Benchmarks/Horizontal Alignment

- MA.912.AR.2.5
- MA.912.AR.5.3, MA.912.AR.5.5
- MA.912.DP.2.4, MA.912.DP.2.9
- MA.912.F.1.6, MA.912.F.1.8

Terms from the K-12 Glossary

- Exponential function
- Simple interest
- Slope

Vertical Alignment

Previous Benchmarks

- MA.7.AR.3.1
- MA.912.NSO.1.1

Next Benchmarks

- MA.912.FL.4.3, MA.912.FL.4.4

Purpose and Instructional Strategies

In Algebra I, students explained the relationship between simple interest and linear growth and the relationship between compound interest and exponential growth. In Math for College Liberal Arts, students extend this to include continuously compounded interest. In other courses, students will describe the advantages and disadvantages of various financial and investment plans.

- In MA.912.FL.3.1 and MA.912.FL.3.2, students became familiar with simple and compound interest and how to use the formulas for each to solve real-world problems. In this benchmark, students will make connections between simple interest and linear growth as well as between compound interest and exponential growth, extending this connection to include continuously compounded interest. To help students discover this relationship by examining patterns and structure (*MTR.5.1*), consider guiding them to create a table.
    - For example, Brian, Paul and Abby receive $1000 cash in graduation gifts from family and friends. They each decide to invest their money in an investment account. Brian's investment earns 10% in *simple* interest annually. Paul's

investment earns 10% in interest *compounded* annually. Abby's investment earns 10% in interest *compounded continuously*. Guide students to identify the interest formulas and use them to create the table below to compare the growth of their investments over time.

  - Brian's Total Value would be represented by $A_B = 1000(1 + 0.1t)$.
  - Paul's Total Value would be represented by $A_P = 1000(1 + 0.1)^t$.
  - Abby's Total Value would be represented by $A_A = 1000e^{(0.1t)}$.

Years Invested (*t*)	**Total Value of Brian's Investment (A_B)**	**Total Value of Paul's Investment (A_P)**	**Total Value of Abby's Investment (A_A)**
1	1,100	1,100	1,105.17
2	1,200	1,210	1,221.40
3	1,300	1,331	1,349.86
4	1,400	1,464.10	1,491.82
5	1,500	1,610.51	1,648.72
10	2,000	2,593.74	2,718.28
15	2,500	4,177.25	4,481.69
20	3,000	6,727.50	7,389.06
30	4,000	17,449.40	20,085.54
50	6,000	117,390.90	148,413.16

  - Once completed, ask students what relationships they observe in the behavior of each of the three investments. Student are guided to discover that Brian's investment exhibits linear growth, while Paul's and Abby's investment shows exponential growth (MA.912.F.1.8).
  - Solidify this understanding by having students graph the three functions that represent the total value of the investments (*MTR.2.1*).
  - Once students make the discovery, begin a conversation with them about which type of interest would be more advantageous for long-term investments. Take this opportunity to make connections to MA.912.F.1.6 (i.e., asking students questions like "What if Brian received $10,000 in gifts?" or "Would the simple interest be a better investment tool?").
  - The expectation for this benchmark is for students to explain *why* these relationships occur. Be sure to discuss how the variation of years is used as a factor in the simple interest formula and as an exponent in the compounded and continuously compounded interest formulas.

- Problems related to this benchmark may ask students for the interest earned over a period of time while others may ask for the account balance or total value of the investment over a period of time.
- Instruction should include real-world problems on simple, compounded interest and continuously compounded interest in connection with MA.912.AR.5.5.

Common Misconceptions or Errors

- Students may not understand when to use an exponential function based on a table of values or written description.
- Students may interpret any relationship that increases/decreases at a non-constant rate as being an exponential relationship. Have these students verify an exponential relationship by looking for common ratios. If they are interpreting a written description, direct them to make a sample table of values from the context to examine.
- When forming compound interest equations, students sometimes forget to convert the interest rate from a percent value to a decimal value before substituting it into the formula.
- Some students may miss finding the interest earned over a period of time and may always calculate total interest for simple interest problems and total value for compound interest problems. In these cases, point students back to the wording of the problem and help them assess the reasonableness of their answers (*MTR.6.1*) in context.
- Students may assume continuously compounding accounts will have a substantially larger returns, when in fact, the difference in the total interest earned through continuous compounding is not very high when compared to traditional compounding periods.

Instructional Tasks

Instructional Task 1 (MTR.3.1, MTR.4.1, MTR.5.1, MTR.7.1)

Kyla invests in a savings account that applies simple interest.

Part A. How will her investment grow, linearly or exponentially? Justify your answer.

Part B. If Kyla invests $725 and earns an annual rate of 4.2%, write an equation that would represent the total amount she would have at the end of each year.

Part C. How long will it take for her initial investment to double?

Part D. If instead the savings account had interest at the same rate but was compounded continuously, how much money would she have after the amount of time found in Part C?

Instructional Items

Instructional Item 1

Mikey invests in a savings account that applies compound interest weekly. How will his investment grow – linearly or exponentially? Justify your answer.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.GR.1 *Prove and apply geometric theorems to solve problems.*

MA.912.GR.1.6

Benchmark

MA.912.GR.1.6 Solve mathematical and real-world problems involving congruence or similarity in two-dimensional figures.

Benchmark Clarifications:
Clarification 1: Instruction includes demonstrating that two-dimensional figures are congruent or similar based on given information.

Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.2.4
- MA.912.GR.4.3
- MA.912.T.1.2

Terms from the K-12 Glossary

- Congruent
- Similarity

Vertical Alignment

Previous Benchmarks

- MA.912.GR.1.1, MA.912.GR.1.2, MA.912.GR.1.4, MA.912.GR.1.5
- MA.912.GR.2.3, MA.912.GR.2.6, MA.912.GR.2.8
- MA.912.GR.3.3

Next Benchmarks

- MA.912.T.2.2, MA.912.T.2.3

Purpose and Instructional Strategies

In Geometry, students established congruence and similarity using criteria from Euclidean geometry and using rigid transformations. In Math for College Liberal Arts, students extend this knowledge to solve mathematical and real-world problems involving congruence or similarity in two-dimensional figures. In other courses, students will use similar triangles to develop trigonometry related to the unit circle.

- Instruction connects students' prior knowledge on proving theorems about triangle congruence to other polygons, as many polygons can be broken down into triangles.
  - This will translate to preparation for trigonometry in future courses because trigonometry is based on triangle relationships.
- Problem types include using congruence and similarity criteria to determine whether two triangles are congruent or similar, and using the definition of congruence and similarity to find missing angle measures and side lengths.
- Instruction includes discussing the definitions of congruent polygons and similar polygons based on corresponding parts. Students should understand that if a problem involves polygons, they will have to use the definitions of congruent and similar (there are no congruence or similarity criteria for polygons) to show the polygons are congruent or similar and use this information to solve the task (*MTR.4.1*).
  - When two polygons are congruent, corresponding sides and corresponding angles

are congruent.
    - When two polygons are similar, corresponding angles are congruent and corresponding sides are in proportion.
- Instruction includes the use of constructions, patty paper, technology, etc. to show and verify cases of congruence and similarity.

Common Misconceptions or Errors

- Students may have difficulty separating overlapping similar or congruent triangles. To help address this misconception, have students draw the two triangles separately with their corresponding known measures and lengths.
- Students may expect that there are congruence or similarity criteria for non-triangular polygons that are like the criteria for triangles. To help address this misconception, have students create examples in which two quadrilaterals are not congruent even though they have corresponding congruent sides to see that there is no Side-Side-Side-Side congruence criterion for quadrilaterals.
- Students may struggle with setting up proportional relationships correctly.

Instructional Tasks

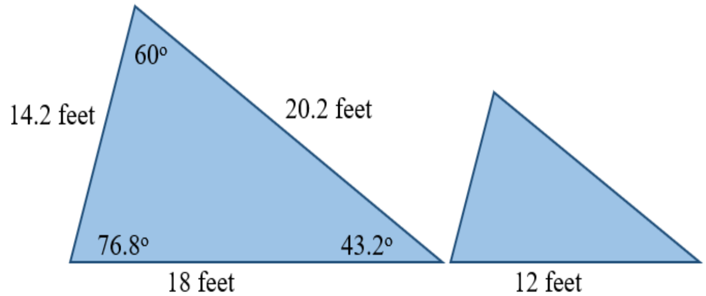
Instructional Task 1 (MTR.4.1)

Moise explains that all rectangles are similar because they have congruent, corresponding angles of 90°.

Part A. Determine if Moise is correct and justify your reasoning.
Part B. Provide an example and a counterexample.
Part C. How can Moise change his statement so that it would be correct?

Instructional Task 2 (MTR.6.1, MTR.7.1)

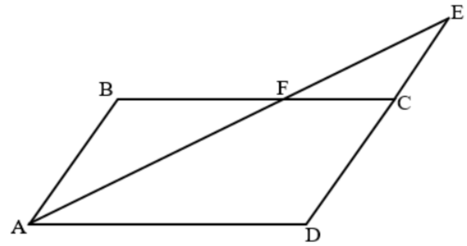
A designer is using similar triangles to create a sculpture in the town square. She began with the largest triangle, as shown below, and the base of the second triangle.



Part A. Without computation, what would be a reasonable estimate for the other two sides of the second triangle be?
Part B. What do the measurements of the other two sides of the second triangle need to be in order to ensure similar figures?
Part C. If a third triangle was added to the sculpture, what scale factor should she use? Why?

Instructional Task 3 (MTR.3.1)

In the figure below, $ABCD$ is a parallelogram. The length of segment AB is 7 units, the length of segment BF is 8 units, the length of segment FC is 3 units and the length of segment AF is 13 units.

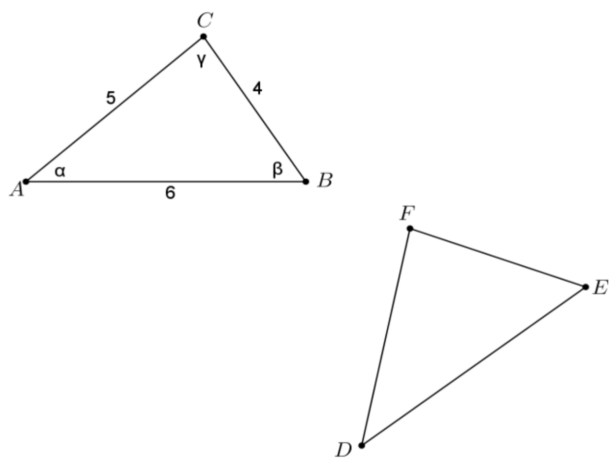


Part A. Identify a pair of similar triangles in the diagram.
Part B. Explain why the triangles you named are similar.
Part C. Find the length of FE and the length of CE. Show all of your work and leave your answers exact.

Instructional Items

Instructional Item 1

$\Delta ABC \cong \Delta DEF$. The lengths of the sides and the measures of the angles of ΔABC are shown in the diagram.



Determine the lengths of the sides and the measures of the angles of ΔDEF.

Instructional Item 2

The basketball coach is refurbishing the outdoor courts at his school and is wondering if the goals are at the regulation height. The regulation height is 10 feet, measured from the ground to the rim. One afternoon the gym teacher, who is 6 feet tall, measured his own shadow at 5 feet long. He measured the shadow of the basketball goal (to the rim) as 8 feet long. Use this information to determine if the basketball goal is at the regulation height.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.GR.2 *Apply properties of transformations to describe congruence or similarity.*

MA.912.GR.2.4

Benchmark

MA.912.GR.2.4 Determine symmetries of reflection, symmetries of rotation and symmetries of translation of a geometric figure.

Benchmark Clarifications:
Clarification 1: Instruction includes determining the order of each symmetry.
Clarification 2: Instruction includes the connection between tessellations of the plane and symmetries of translations.

Connecting Benchmarks/Horizontal Alignment

- MA.912.AR.2.5
- MA.912.AR.5.6
- MA.912.GR.1.6

Terms from the K-12 Glossary

- Reflection
- Rotation
- Translation

Vertical Alignment

Previous Benchmarks

- MA.912.GR.2.1, MA.912.GR.2.2, MA.912.GR.2.3, MA.912.GR.2.5

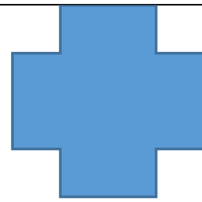
Next Benchmarks

- MA.912.T.2
- MA.912.T.3

Purpose and Instructional Strategies

Symmetries of reflection were introduced in the elementary grades through lines of symmetry. In Geometry, students studied other types of symmetries coming from rigid transformations that map a polygon onto itself, and they determined the number of times such a transformation must be applied before each point in the polygon is mapped to itself. In Math for College Liberal Arts, students determine symmetries of reflection, symmetries of rotation and symmetries of translation of a geometric figure.

- Instruction includes multiple opportunities for students to explore symmetries using both physical exploration (transparencies, mirrors or patty paper) and virtual exploration, when possible.
- Instruction includes using a variety of shapes (mathematical and real world) to explore the reflection symmetry and rotational symmetry of the shapes. Include identifying the lines of symmetry, the order of symmetry and the angle of rotation that will map the figure onto itself.
- The order of symmetry is the smallest (nonzero) number of times that you must apply the corresponding transformation to map each point of the figure onto itself.
- The order of rotational symmetry is the number of times the figure maps onto itself as it rotates through 360° about the figure’s center. Instruction includes identifying the angles of rotation when determining symmetries of rotation.
    - For example, the order of rotational symmetry for a regular hexagon is 6 with the angle of rotation of 60°.
    - For example, the order of rotational symmetry for an isosceles trapezoid is 0.
    - For example, the figure below has an order of 4 with angle of rotation of 90°.



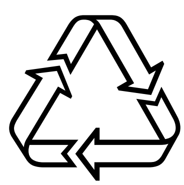
- The order of a translational symmetry is infinite because no matter how many times one applies it, no point gets mapped onto itself. Translational symmetry results from mapping a figure onto itself by moving it a certain distance in a certain direction. Show students tessellations (covering of a plane using one or more geometric shapes with no overlaps and no gaps), or have them create them, and discuss the translational symmetry in the tessellation (*MTR.5.1*).
- Depending on a student's pathway, instruction includes making the connection to symmetry as a key feature of the graphs of polynomial and trigonometric functions and the importance of applying such functions in the real world.
- Problem types include using symmetries to classify geometric figures.
  - For example, given that a two-dimensional figure has one line of symmetry through midpoints of the two parallel sides, no rotational symmetry and no point symmetry, students can deduce that this figure is a trapezoid.

Common Misconceptions or Errors

- Students may not make the connections to the idea of an infinite pattern (wallpaper pattern, border pattern, etc.) when working with only a portion of that pattern. If students arrive at an order for a translational symmetry that is not infinite, ask them if their answer would be different if the pattern continued.
- Students may have difficulty distinguishing between mapping a figure onto itself and mapping every point of the figure onto itself. To help address this misconception, have students highlight a particular point and observe how it is affected by one application of a transformation that maps the figure onto itself.

Instructional Tasks

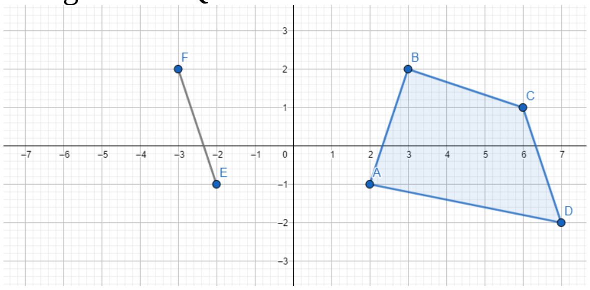
Instructional Task 1 (MTR.4.1, MTR.5.1)
Use the figures below to answer the questions.

Part A. Draw the lines of symmetry, if any, on each figure.
Part B. What is the order of symmetry for reflections?
Part C. Determine the order of rotational symmetry for each figure. Discuss with a partner how you determined each.

Instructional Task 2 (MTR.3.1, MTR.5.1,)

Given Quadrilateral $ABCD$ and $\overline{EF}$ below, where would points G and H need to lie so that Quadrilateral $ABCD$ is congruent to Quadrilateral $EFGH$?



Instructional Items

Instructional Item 1

Part A. Which capital letters that are vowels of the English alphabet have a line of symmetry?

Part B. Do any of the vowels identified from Part A also have rotational symmetry?

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.GR.4 *Use geometric measurement and dimensions to solve problems.*

MA.912.GR.4.3

Benchmark

MA.912.GR.4.3	Extend previous understanding of scale drawings and scale factors to determine how dilations affect the area of two-dimensional figures and the surface area or volume of three-dimensional figures.

Example: Mike is having a graduation party and wants to make sure he has enough pizza. Which option would provide more pizza for his guests: one 12-inch pizza or three 6-inch pizzas?

Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.1.6

Terms from the K-12 Glossary

- Area
- Scale factor
- Scale model

Vertical Alignment

Previous Benchmarks

- MA.912.GR.4.3

Next Benchmarks

Purpose and Instructional Strategies

In Geometry, students used their knowledge of scale drawing and scale factors to learn about how changes in the dimensions of a figure due to a dilation will affect the area of two-dimensional figures and the surface area or volume of three-dimensional figures in a way they can predict. In Math for College Liberal Arts, students continue to work on how dilations affect the area of two-dimensional figures and the surface area or volume of three-dimensional figures.

- Instruction includes exploring the effect of changing the dimensions of two-dimensional and three-dimensional figures using different factors. It may be helpful to begin exploring through specific problems working with a table of values or with algebraic formulas.
  - For example, have students explore what happens to the area of a rectangle if the height is doubled and the length is tripled. Additionally, have them explore what happens to the volume of a cylinder if the height is multiplied by 0.5 and the radius is multiplied by 4.
- Instruction includes reviewing that the area of the image of a dilation with scale factor k is k^2 times the area of the pre-image for any two-dimensional figure (as this was done in grade 7).
- Instruction includes the student understanding that the surface area of the image of a dilation with scale factor k is k^2 times the surface area of the pre-image, and the volume of the image of a dilation with scale factor k is k^3 times the volume of the pre-image for any three-dimensional figure.

Common Misconceptions or Errors

- Students may multiply the area, surface area or volume by the scale factor instead of thinking about the multiple dimensions.
- Students may believe the scale factor has the same effect on surface area and volume. To help address this, discuss the effects on surface area using two-dimensional nets of simple figures and then compare to the effects on volumes.
- Students may incorrectly apply units of measure when problem solving.
  - For example, students may use cubic units to indicate surface area instead of square units.

Instructional Tasks

Instructional Task 1 (MTR.4.1, MTR.5.1)

Use the table below to answer the following questions.

Original Cylinder	**Dilation with scale factor *k***	**New Surface Area**	**New Volume**
Radius of the base is 5 inches Height of the cylinder is 12 inches Surface area = _____ sq. inches Volume = _____ cubic inches	$k = 2$		
	$k = 3$		
	$k = \frac{1}{2}$		

Part A. Determine the surface area and volume of the given cylinder.
Part B. Given the three different dilations determine the new surface areas and volumes.
Part C. Compare each of the new surface areas to the original surface area. What do you notice?

Part D. Compare each of the new volumes to the original volume. What do you notice?
Part E. Predict the surface area and volume of the cylinder resulting from a dilation with a scale factor of 5. Explain the method you chose.

Instructional Task 2 (MTR.4.1, MTR.5.1)

Sally and Nancy were solving problems involving square pyramids. Sally's square pyramid had a side length of 4 inches. The height of her pyramid was 8 inches. Nancy's square pyramid had a side length of 8 inches and a height of 16 inches.

Part A. How do the volumes of each of the two square pyramids compare?
Part B. What is the scale factor between Sally's and Nancy's square pyramids?
Part C. Does your answer change if you know that Nancy made her pyramid first?

Instructional Items

Instructional Item 1

A traditional snow cone has a height of 13 centimeters and a radius of 4 centimeters. SnowCone4U is creating a three-dimensional replica for the top of the store. The scale factor is 8 times larger. If SnowCone4U is purchasing plastic to model the cone, how much plastic in square centimeters is needed to cover the cone if it is enclosed?

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.GR.4.4

Benchmark

MA.912.GR.4.4 Solve mathematical and real-world problems involving the area of two-dimensional figures.

Example: A town has 23 city blocks, each of which has dimensions of 1 quarter mile by 1 quarter mile, and there are 4500 people in the town. What is the population density of the town?

Benchmark Clarifications:

Clarification 1: Instruction includes concepts of population density based on area.

Connecting Benchmarks/Horizontal Alignment

- MA.912.T.1.2

Terms from the K-12 Glossary

- Area
- Circle
- Rectangle
- Square
- Trapezoid
- Triangle

Vertical Alignment

Previous Benchmarks
- MA.912.AR.2.1
- MA.912.GR.3.4

Next Benchmarks
- MA.912.C.5

Purpose and Instructional Strategies

In Geometry, students solved mathematical and real-world problems involving the area of two-dimensional figures. In Math for College Liberal Arts, students continue to work on solving real-world problems involving the area of two-dimensional figures. In other courses, students will use integrals to connect the concept of area to many other real-world and mathematical contexts.

- Instruction includes reviewing units and conversions within and across different measurement systems.
- Instruction includes discussion around dimensions. Students should understand that perimeter/circumference is a one-dimensional measure. Area is a two-dimensional measure. Volume is a three-dimensional measure.
- Instruction includes the idea that formulas build on each other. The area of a rectangle formula is the basis for all other polygon area formulas. As figures become more complex, the formulas also do. However, they are based on the concept that the area is found by multiplying the two perpendicular dimensions to determine the number of square units in a figure.
- Instruction includes making connections to learning from previous coursework. Below describes some ways in which students may have developed an understanding of each formula for the two-dimensional figure. For mastery of this benchmark, there is no expectation to memorize formulas for area of two-dimensional figures.
    - Rectangle
      Area is a result of length times width or base times height $A = lw$ or $A = bh$.
    - Parallelogram
      Through the process of dissection, students explored how cutting up shapes into smaller pieces and moved them around form new shapes. Area is the result of length times width or base times height $A = lw$ or $A = bh$.
    - Triangle
      The area of a triangle can be derived from the area of the rectangle. Students explored this concept by using dissection strategies to show that the area of the triangle is exactly half the area of a rectangle with the same base and height $A = \frac{1}{2}bh$.
    - Trapezoid
      The area of trapezoid showcases one of the most unique formulas of all basic polygons. Students learned different ways to derive the formula through dissection strategies to show that the area of a trapezoid is half the height multiplied by the sum of the bases $A = \frac{1}{2}h(b_1 + b_2)$.
    - Regular Polygons
      A regular polygon refers to a specific type of closed figure that has all equal sides, angles, and the maximum lines of symmetry for that particular number of sides. Regular polygons can be inscribed in a circle and through this process the apothem is found and defined as the perpendicular distance from the center to a

side of the regular polygon. Through this exploration using special right triangles, students discover the area of a regular polygon is half the perimeter times the apothem. This leads to the formula $A = \frac{1}{2}Pa$, where P is the perimeter and a is the length of the apothem.

  - Circle
    Students explored the area of a circle by using multiple dissections to create a shape that looks like a parallelogram. The dimensions of the parallelogram show the height approaches the radius of the circle, and the length of the base is half the circumference of the circle $A = \pi r^2$.
  - Area of a Circle Sector
    A sector of a circle is a part of the circle formed by two radii and the circumference. The area of a sector is found by multiplying the percentage of the central angle $\left(\frac{x°}{360°}, \text{where } x \text{ is the measure of the central angle}\right)$ by the area of the circle $A = \frac{x°}{360°}\pi r^2$.

- Instruction includes discussing the convenience of answering with exact values (e.g., the simplest radical form or in terms of pi) with approximations (e.g., rounding to the nearest tenth or hundredth or using $3.14, \frac{22}{7}$ or other approximations for pi). It is also important to explore the consequences of rounding partial answers on the accuracy or precision of the final answer, especially when working in real-world contexts.
- Instruction includes exploring a variety of real-world situations where finding the area is relevant for different purposes. Problem types include components like percentages, cost and budget, constraints, comparisons and others.
- Problem types include finding missing dimensions given the area of a two-dimensional figure or finding the area of composite figures.
- Instruction includes the understanding that density is the quantity per unit of volume, area or length. The population density based on area is the quotient of the total population and the total area.
  - For example, the population of Florida in 2010 was 18,801,310 and the land area was 53,625 square miles. This represents an approximate population density of 351 people per square mile.
- Have students practice finding population density or the total population, given the dimensions of a two-dimensional figure. That is, part of their work includes finding the area based on the dimensions *(MTR.7.1).*

Common Misconceptions or Errors

- Students may incorrectly identify a side length as a height rather than using the perpendicular distance between the bases.
- Students may not properly locate the height or base(s) when using figures in various orientations.
- Students may invert the terms radius and diameter.
- Students may incorrectly believe pi is a variable rather than a constant for every circle.
- Students may incorrectly double the radius rather than squaring it when finding the area of a circle.
- Students may not be careful with units of measurement involving area, particularly when converting from one unit to another.

- For example, since there are 3 feet in a yard, a student may not correctly conclude that there are 9 square feet in a square yard.

Instructional Tasks

Instructional Task 1 (MTR.3.1, MTR.7.1)

Abbie is debating about which pizza is a better buy: a large or medium pizza. The large pizza costs $15.00 and has a diameter of 18 inches. A medium pizza costs $7.50 and has a diameter of 14 inches.

Part A. What is the area of the large pizza?
Part B. What is the area of the medium pizza?
Part C. What is the price per square inch for the large pizza?
Part D. What is the price per square inch for the medium pizza?
Part E. What is the better buy: a large pizza or medium pizza? Explain your answer.

Instructional Task 2 (MTR.1.1, MTR.7.1)

A rectangular floor measures 18 feet by 24 feet. What will it cost to carpet the floor if the carpet costs $56 per square meter?

Part A. What questions still need to be answered to approach this problem?
Part B. If 1 foot is approximately 0.3 meter, what is the cost to install the carpet?

Instructional Task 3 (MTR.4.1, MTR.7.1)

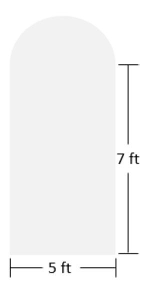
In 2019, the population of St. Lucie County was 328,297, and the population of Sarasota County was 433,742. The area of Sarasota County is 752 square miles, while the area of St. Lucie County is 688 square miles.

Part A. Which county has a higher population density?
Part B. If the physical shape of St. Lucie County is a right trapezoid with the non-perpendicular leg of the trapezoid being approximately 21 miles long, what is one set of possible dimensions of the bases and height of the county?
Part C. How do your dimensions compare to a partner? What steps did you take to solve this problem?

Instructional Items

Instructional Item 1

A window with a half-circle and a rectangle is part of the blueprint of Sarah's new house. She is ordering the glass that will be installed in the window. What is the area of the glass required to cover the window?



**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.GR.4.5

Benchmark

MA.912.GR.4.5 Solve mathematical and real-world problems involving the volume of three-dimensional figures limited to cylinders, pyramids, prisms, cones and spheres.

Example: A cylindrical swimming pool is filled with water and has a diameter of 10 feet and height of 4 feet. If water weighs 62.4 pounds per cubic foot, what is the total weight of the water in a full tank to the nearest pound?

Benchmark Clarifications:
Clarification 1: Instruction includes concepts of density based on volume.
Clarification 2: Instruction includes using Cavalieri's Principle to give informal arguments about the formulas for the volumes of right and non-right cylinders, pyramids, prisms and cones.

Connecting Benchmarks/Horizontal Alignment

- MA.912.T.1.2

Terms from the K-12 Glossary

- Cone
- Cylinder
- Prism (right)
- Pyramid
- Sphere

Vertical Alignment

Previous Benchmarks

- MA.912.AR.2.1

Next Benchmarks

- MA.912.C.5.7

Purpose and Instructional Strategies

In Geometry, students solved mathematical and real-world problems involving the volume of three-dimensional figures limited to cylinders, pyramids, prisms, cones and spheres. In Math for College Liberal Arts, students continue to work on solving real-world problems involving the volume of three-dimensional figures. In other courses, students learn more advanced methods for calculating volume.

- Instruction includes reviewing units and conversions within and across different measurement systems.
- Instruction includes the idea that volume can be found by using the area of the base and stacking the base as many units as the height.
- Instruction includes discussion around dimensions. Students should understand that perimeter/circumference is a one-dimensional measure. Area is a two-dimensional measure. Volume is a three-dimensional measure.
- Instruction includes making connections to learning from previous coursework. Below describes some ways in which students may have developed an understanding of each formula for the three-dimensional figure. For mastery of this benchmark, there is no expectation to memorize formulas for volume of three-dimensional figures.
    - Prism
      To find the volume, students explored the stacking principle in which infinitesimal cross-sections of the figure or areas of the base of the prism stack on each other to fill the volume of the shape. This develops the formula of $V = Bh$, where B represents the area of the base and h represents the height of the prism.
    - Cylinder

This three-dimensional figure has the characteristics of a prism but with a circular base. The stacking technique works for the volume of a cylinder. The formula of $V = Bh$ is still used, but the B can be rewritten as πr^2. So the volume of a cylinder can also be represented as $V = \pi r^2 h$.

- Pyramid
  To derive this formula, students exploration would include transferring the pyramid's contents into a prism with the same base and height. Students discover that the contents of three pyramids will fill the prism, thus making the ratio between the volume of the pyramid to the volume of the prism is $\frac{1}{3}$. The formula for the volume of a pyramid is $V = \frac{1}{3}Bh$.
- Cone
  Students' exploration to find the volume formula for the cone follows the same pattern as the pyramid and the prism. The formula of $V = \frac{1}{3}Bh$ is still used, but the B can be rewritten as πr^2. So the volume of a cone can also be represented as $V = \frac{1}{3}\pi r^2 h$.

- Sphere
  The formula for the volume of a sphere is a little more difficult to visualize. The radius represents all three dimensions. Exploration to find the volume formula of a sphere can be demonstrated by filling the sphere with the contents of two cones with the same radius measure resulting in the formula of $2\left(\frac{1}{3}\pi r^2 h\right)$. The height of the cone is $2r$, representing the diameter of the sphere. By substitution, the formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$.
- Cavalieri's Principle
  Bonaventura Cavalieri investigated the ideas of the stacking principle and developed Cavalieri's Principle, which states that if in two solids of equal height, the cross-sections made by planes parallel to and at the same distance from their respective bases are always equal, then the volumes of the two solids are equal (*MTR.5.1).*
  - For example, have students compare the volume of two stacks of pennies of the same height, one organized in a straight column and the other one, one penny on top of the other, but in a slanted stack. Discuss the shape of their cross-sections at the same height and what happens with their volumes.
  - For example, have students discuss how this principle is applied in the calculation of volumes of non-right (oblique) three-dimensional figures.
  - For example, have students discuss how this principle can be used to find the volume of a non-right cylinder given a right cylinder with the same height and same cross-sections *(MTR.4.1).*

- Instruction includes discussing the convenience of answering with exact values (e.g., the simplest radical form or in terms of pi) with approximations (e.g., rounding to the nearest tenth or hundredth or using $3.14, \frac{22}{7}$ or other approximations for pi). It is also important to explore the consequences of rounding partial answers on the accuracy or precision of the final answer, especially when working in real-world contexts.

- Instruction includes exploring a variety of real-world situations where finding the volume or volume density is relevant for different purposes. Problem types include components like percentages, cost and budget, constraints, comparisons, British thermal unit (BTU), nutrition (e.g., calories per cup), moisture content (e.g., ounces of water in a gallon of honey) or others.
- Problem types include finding missing dimensions given the volume of a three-dimensional figure or finding the volume of composite figures.
- Instruction includes the understanding that density is the quantity per unit of volume, area or length (e.g., the population density of fish in a spherical aquarium or density of salt in a bucket of water).
    - For example, Molly has 20 pounds of clay to make a lawn ornament in the form of a solid sphere. If the radius of a sphere is 4 inches, how much does the lawn ornament weigh per cubic inch?
- Have students practice finding the population or material density or the total population or material amount, given the dimensions of a three-dimensional figure. That is, part of their work includes finding the volume based on the dimensions (*MTR.7.1).*

Common Misconceptions or Errors

- Students may not properly locate the height or base(s) when using figures in various orientations.
- Students may invert the terms radius and diameter.
- Students may incorrectly believe pi is a variable rather than a constant for every circle.
- Students may incorrectly apply surface area and volume concepts.
- Students may not be careful with units of measurement involving volume, particularly when converting from one unit to another.
    - For example, since there are 3 feet in one yard, a student may not correctly conclude that there are 27 cubic feet in a cubic yard.

Instructional Tasks

Instructional Task 1

Alexander has 5 cube-shaped samples of wood. Five of the labels on the cubes have fallen off, so he doesn't know what type of wood each is. The table below provides the densities for different types of wood. Help Alexander determine which type of wood each sample is.

Type of Wood	Density (g/cm^3)
Ash	0.638
Birch	0.601
Elm	0.554
Hemlock	0.431
Maple	0.676
Oak	0.711
Pine	0.373

Part A. What measurements does Alexander need in order to determine the type of wood each sample is?

Part B. Complete the table below to determine each type of wood.

Sample	Edge Length	Mass	Type of Wood
Sample 1	6 cm	119.7 grams	
Sample 2	4 cm	40.8 grams	
Sample 3	5 cm	88.9 grams	
Sample 4	3 cm	10.1 grams	
Sample 5	7 cm	231.9 grams	

Part C. You have a cube of birch wood that is 5 cm long. What is the mass of the birch wood?

Instructional Items

Instructional Item 1

Tari is building a swimming pool with the dimensions of 90 feet long, 60 feet wide, and 6 feet deep. The company that hauls the dirt away charges $35 per 10 cubic yards of dirt. How much will it cost Tari to have all the dirt hauled away?

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.GR.4.6

Benchmark

MA.912.GR.4.6	Solve mathematical and real-world problems involving the surface area of three-dimensional figures limited to cylinders, pyramids, prisms, cones and spheres.

Connecting Benchmarks/Horizontal Alignment

- MA.912.T.1.2

Terms from the K-12 Glossary

- Circle
- Cone
- Cylinder
- Prism
- Pyramid
- Rectangle
- Sphere
- Square
- Trapezoid
- Triangle

Vertical Alignment

Previous Benchmarks

- MA.912.GR.4.6

Next Benchmarks

Purpose and Instructional Strategies

In Geometry, students solved mathematical and real-world problems involving the surface area of three-dimensional figures limited to cylinders, pyramids, prisms, cones and spheres. In Math for College Liberal Arts, students will continue this practice.

- Instruction includes reviewing units and conversions within and across different measurement systems.
- Instruction includes making connections to learning from previous coursework. Below describes some ways in which students may have developed an understanding of each formula for the three-dimensional figure. For mastery of this benchmark, there is no expectation to memorize formulas for surface area of three-dimensional figures.
    - Cylinder
      The surface of a cylinder is made up of 2 circles and a rectangle. The area of each circular base is represented as $B = \pi r^2$ and the area of the rectangle is the circumference of the circular base, P, times the height, h, or $Ph = 2\pi rh$. The formula for the surface area of a cylinder can be represented as $SA = 2B + Ph$, $SA = 2(\pi r^2) + 2\pi rh$ or $SA = 2\pi r(r + h)$.
    - Pyramid
      The surface of a pyramid is made up of a polygonal base with a triangular face for every side of the base. The B represents the area of the polygonal base, the A represents the area of each face. The formula for the surface area of a pyramid can be written as the sum of the area of the base and the area of each triangular face $SA = B + A(\text{each face})$.

- Prism
  The surface of a prism is made up of two polygon bases and a rectangular face for every side of the base. The B represents the area of the polygonal base, the P represents the perimeter of the base, and h represents the height of the prism. The formula for the surface area of a prism can be written as the sum of the area of the two bases and the perimeter times the height of the prism $SA = 2B + Ph$.
- Cone
  The surface of a cone is made up of a circular base and a curved surface. The surface area of a cone is the result of combining the area of the base, a circle with radius, r, and circumference C, and the area of the curved surface, a circular sector with radius, h_s, which is the slant height of the cone and arc length C. The slant height h_s is $\sqrt{h^2 + r^2}$, where h is the height of the cone. The area of the circle is πr^2 and the area of the circular sector is $\frac{1}{2}Ch_s$, which is equivalent to $\pi r h_s$. Therefore, given a cone with radius, r, and height, h, its surface area is $SA = B + \pi r h_s$.
- Sphere
  Since deriving the surface area of a sphere requires Calculus, students will not be able to explore its formula. A sphere's surface area can use the formula $SA = 4\pi r^2$.

- Instruction includes discussion around dimensions. Students should understand that perimeter/circumference is a one-dimensional measure. Area is a two-dimensional measure. Volume is a three-dimensional measure.
- Instruction includes exploring the surface area of cylinders, pyramids and prisms as the result of combining areas of triangles, rectangles and circles (and when needed other polygons). Students should understand the similarities and differences between lateral area and surface area (*MTR.2.1)*.
- Instruction includes discussing the convenience of answering with exact values (e.g., the simplest radical form or in terms of pi) with approximations (e.g., rounding to the nearest tenth or hundredth or using $3.14, \frac{22}{7}$ or other approximations for pi). It is also important to explore the consequences of rounding partial answers on the accuracy or precision of the final answer, especially when working in real-world contexts.
- Instruction includes exploring a variety of real-world situations where finding the surface area is relevant for different purposes. Problem types include components such as percentages, cost and budget, constraints, comparisons or others.
- Problem types include finding missing dimensions given the surface area of a three-dimensional figure, finding the surface area of composite figures or determining which face to include in calculations within real-world context (e.g., the surface area required to paint a house, the surface area that will be covered by a label in a soup can).

Common Misconceptions or Errors

- Students may inadvertently mistake slant heights for perpendicular heights within figures.
- Students may inadvertently mistake volume and surface area problems.
- Students may have trouble working with formulas by making incorrect substitutions or incorrect use of the order of operations.

Instructional Tasks

Instructional Task 1 (MTR.4.1)

An engineer has been asked to redesign a cylindrical can with a diameter of 4.25 inches. To make the can, approximately 153 cubic inches of aluminum is needed.

Part A. What is the height of the cylindrical can?

Part B. If the local grocery store plans to put multiple cans on a shelf that is 84 inches long, 7 inches high and 12 inches deep, how many cans could fit on a single shelf?

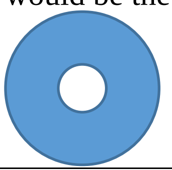
Instructional Items

Instructional Item 1

A right cylinder has a height of 24.3 feet and radius of 18.2 feet.

Part A. What is the surface area of the right cylinder?

Part B. If the right cylinder has a hole (see cross section below; note: not drawn to scale) with diameter of 3.2 feet, what would be the surface area of the cylinder?



**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

Trigonometry

MA.912.T.1 *Define and use trigonometric ratios, identities or functions to solve problems.*

MA.912.T.1.2

Benchmark

MA.912.T.1.2 Solve mathematical and real-world problems involving right triangles using trigonometric ratios and the Pythagorean Theorem.

Benchmark Clarifications:
Clarification 1: Instruction includes procedural fluency with the relationships of side lengths in special right triangles having angle measures of 30°-60°-90° and 45°-45°-90°.

Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.4.6

Terms from the K-12 Glossary

- Angle
- Equilateral triangle
- Hypotenuse
- Isosceles triangle
- Right triangle

Vertical Alignment

Previous Benchmarks

- MA.912.T.1.2

Next Benchmarks

- MA.912.T.1.5, MA.912.T.1.6, MA.912.T.1.7, MA.912.T.1.8
- MA.912.T.2
- MA.912.T.3

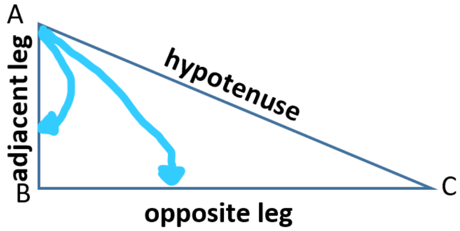
Purpose and Instructional Strategies

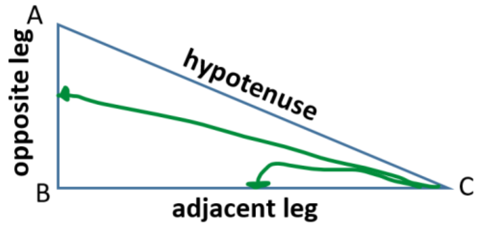
In Geometry, students solved mathematical and real-world problems involving right triangles using trigonometric ratios and the Pythagorean Theorem. In Math for College Liberal Arts, students continue this work. In other courses, students will extend this knowledge to solve more difficult problems with right triangles and extend the concept of trigonometric ratios to trigonometric functions on the unit circle and the number line.

- Within the Math for College Liberal Arts course, the expectation is to use angle measures given in degrees and not in radians. Additionally, it is not the expectation for students to master the trigonometric ratios of secant, cosecant and cotangent within this course.
- It is customary to use Greek letters to represent angle measures (e.g., $\theta, \alpha, \beta, \gamma$).
- Problem types include cases where information about the side lengths or angle measures of a right triangle is missing and one must use trigonometric ratios, the inverse of trigonometric ratios or Pythagorean Theorem to determine the unknown length(s) or angle measure(s) within a mathematical or real-world context.
- Instruction includes knowing that if either acute angle of a right triangle stays the same, the shape of the triangle does not change even if it is made larger or smaller. This is true due to the properties of similar triangles and the idea that proportionality of the sides

creates ratios for all triangles with similar angles.

- The ratios are defined in terms of the side opposite an acute angle, the side adjacent to the acute angle, and the hypotenuse. It is important to label sides based on the reference angle.



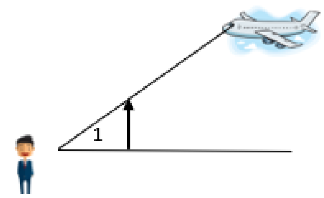


  The hypotenuse will always be opposite the right angle (the longest side). The opposite leg will always be the side of the triangle that does not form the reference angle. The adjacent leg will always be the non-hypotenuse side that forms the reference angle.

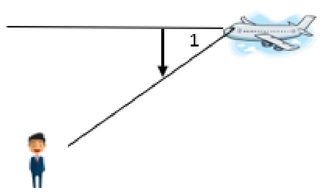
- Trigonometry allows us to link angle size in right triangles to side proportionality. If we know the reference angle, θ, in a right triangle, then there are three fixed ratios of its sides. The converse is also true. If we know the ratio of two sides of a right triangle, then there exists a trigonometric function of the reference angle, θ, having that ratio. Each of these ratios has a special name.
  - Sine (abbreviation sin)
    $\sin\theta = \frac{opposite}{hypotenuse}$
  - Cosine (abbreviation cos)
    $\cos\theta = \frac{adjacent}{hypotenuse}$
  - Tangent (abbreviation tan)
    $\tan\theta = \frac{opposite}{adjacent}$
- Instruction includes how to use a calculator as a tool to determine the ratios and the inverse functions.
- Instruction includes the concept of inverse trigonometric ratios to determine unknown angle measures and how to find these values using technology, including a calculator. Students should have practice using both notations of the inverse trigonometric ratios ($\sin^{-1} A$ or $\arcsin A$, $\cos^{-1} A$ or $\arccos A$, and $\tan^{-1} A$ or $\arctan A$).
- Instruction includes exploring relationships of the side lengths of special right triangles $45° - 45° - 90°$ and $30° - 60° - 90°$.
  - For example, students should realize that the special right triangle $45° - 45° - 90°$ is an isosceles right triangle. Therefore, two of its angle measures and side lengths are equivalent. So, if a side length is x units, then students can use the Pythagorean Theorem to determine that the hypotenuse is $x\sqrt{2}$ units. Additionally, students can make the connection to its trigonometric ratios: $\sin 45° = \frac{1}{\sqrt{2}}$ (or equivalently $\frac{\sqrt{2}}{2}$); $\cos 45° = \frac{1}{\sqrt{2}}$ (or equivalently $\frac{\sqrt{2}}{2}$); and $\tan 45° = 1$.
  - For example, students should realize that the special right triangle $30° - 60° - 90°$ is half of an equilateral triangle. Students can use that knowledge to

determine that the shorter leg is one-half the length of the hypotenuse. So, if the shorter leg is x units and the hypotenuse is $2x$ units, then the students can use the Pythagorean Theorem to determine the other leg is $x\sqrt{3}$ units. Additionally, students can make the connection to its trigonometric ratios such as, $\sin 30° = \frac{1}{2}$; $\cos 30° = \frac{\sqrt{3}}{2}$; and $\tan 30° = \frac{1}{\sqrt{3}}$ (or equivalently $\frac{\sqrt{3}}{3}$).

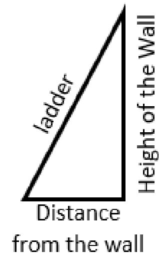
- Instruction should include different contexts to help show how the problems can be applied in the real world. These contexts include, but are not limited to angle of elevation, angle of depression, ladder problems, kite problems, wire problems, and shadows.
  - Angle of Elevation
    The angle of elevation is the reference angle from the position in an upward direction from the horizon.



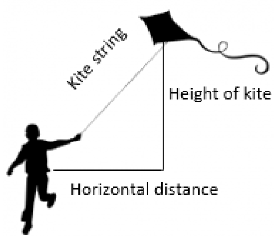
  - Angle of Depression
    The angle of depression is the reference angle from the position in a downward direction from the horizon.



  - Ladder Problems
    The leaning ladder or pole is a common trigonometry problem because of the right angle formed between the wall and the ground. The ladder/pole becomes the hypotenuse of the triangle. Both the angle formed by the ladder and the ground and the angle formed between the ladder and the wall will be referenced in these problem types.



  - Kite Problems
    The kite problem is another trigonometry problem. The length of the string is the hypotenuse. The perpendicular distance from the kite to the ground is the height of the kite and the other leg gets referred to as the ground distance. The only angle used is from the kite flyer to the kite.



- Wire Problems
  A wire can be used to brace or stabilize something. In these problems, the stabilized object becomes the height and the wire or brace becomes the hypotenuse.
- Shadow Problems
  Shadow problems are often used since a shadow can be viewed and measured on the ground. The person or building becomes the height, and the hypotenuse represents the sun's rays. The referenced angles are either the angle formed by the sun's rays and the ground or the angle formed by the sun's rays and the person.

Common Misconceptions or Errors

- Students may inadvertently switch a ratio by not defining the reference angle and the parts associated with the ratio. To address this misconception, students should label the triangle.
- Students may choose the incorrect trigonometric ratio when solving problems.
- Students may incorrectly apply a negative 1 exponent instead of using inverse trigonometric functions.

Instructional Tasks

Instructional Task 1 (MTR.1.1, MTR.7.1)

The Washington Monument is 555 feet high. You are standing one-quarter of a mile from the base of the monument and want to find the angle of elevation to the nearest degree.

Part A. What other information do you need to solve this problem?

Part B. Find the angle of elevation to the nearest degree.

Part C. How would this problem change if you were sitting on top of the Washington Monument looking down at your family that was a half-mile away?

Instructional Items

Instructional Item 1

The Leaning Tower of Pisa is 56.84 meters (m) long. In the 1990s, engineers restored the building so that the angle y changed from 5.5° to 3.99°. To the nearest hundredth of a meter, how much did the restoration change the height of the Leaning Tower of Pisa?

Instructional Item 2

From a point on the level ground 150 feet from the base of a tower, the angle of elevation to the top of the tower is 53.4°. Approximate the height of the tower to the nearest foot.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

Data Analysis & Probability

MA.912.DP.1 *Summarize, represent and interpret categorical and numerical data with one and two variables.*

MA.912.DP.1.1

Benchmark

MA.912.DP.1.1	Given a set of data, select an appropriate method to represent the data, depending on whether it is numerical or categorical data and on whether it is univariate or bivariate.

Benchmark Clarifications:

Clarification 1: Instruction includes discussions regarding the strengths and weaknesses of each data display.

Clarification 2: Numerical univariate includes histograms, stem-and-leaf plots, box plots and line plots; numerical bivariate includes scatter plots and line graphs; categorical univariate includes bar charts, circle graphs, line plots, frequency tables and relative frequency tables; and categorical bivariate includes segmented bar charts, joint frequency tables and joint relative frequency tables.

Clarification 3: Instruction includes the use of appropriate units and labels and, where appropriate, using technology to create data displays.

Connecting Benchmarks/Horizontal Alignment

- MA.912.DP.2.4, MA.912.DP.2.5, MA.912.DP.2.6
- MA.912.DP.3.1, MA.912.DP.3.2, MA.912.DP.3.3

Terms from the K-12 Glossary

- Bar graph
- Bivariate data
- Box plot
- Categorical data
- Circle graph
- Frequency table
- Histogram
- Joint frequency
- Joint relative frequency
- Line graph
- Line plot
- Numerical data
- Scatterplot
- Stem-and-leaf plot

Vertical Alignment

Previous Benchmarks

- MA.912.DP.1.1

Next Benchmarks

- MA.912.DP.2.2
- MA.912.DP.6.6

Purpose and Instructional Strategies

In Algebra I, students displayed univariate data and bivariate numerical data using graphical representations from middle grades and were introduced to bivariate categorical data, which they represent with frequency tables and segmented bar. Additionally, they had to choose an appropriate display when considering each of the four varieties of data. In Math for College Liberal Arts, students continue working on recognizing types of data and methods for representing data. In other courses, students will build upon this foundation as students consider a variety of data distributions in greater detail including normal and Poisson distributions.

- While the benchmark states that students select an appropriate data display, instruction also includes cases where students must create the display.
- This benchmark is closely linked to MA.912.DP.1.2, where students interpret displayed data using key components of the display.
- Numerical univariate is data that consists of one numerical variable, and an important feature of the data is its numerical size or order. Examples include height, weight, age, salary, speed, number of pets, hours of study, etc. Displays include histograms, stem-and-leaf plots, box plots and line plots.
    - Histograms
        - Good for large sets of data.
        - Shows the shape of the distribution to determine symmetry.
        - Data is collected in suitably-sized numerical bins with equal ranges.
        - Because of the bins, only approximate values of individual data points are displayed.
    - Stem-and-Leaf Plots
        - Good for large sets of data.
        - Shows the shape of a data set and each individual data value.
        - Lists exact data values in a compact form.
    - Box Plots
        - Beneficial when large amounts of data are involved or compared. Used for descriptive data analysis.
        - Shows multiple measures of variation and/or spread of data.
        - Shows one measure of central tendency (median).
        - Individual data points are not shown.
        - Presents a 5-number summary of the data (minimum, first quartile, median, third quartile, maximum).
        - Can indicate if a data set is skewed or not, but not the overall shape.
        - Can be used to determine if potential outliers exist.
    - Line Plots (Dot Plots)
        - Used for small to moderate sized data sets in which the numerical values are discrete (often integers, or multiples of ½).
        - Shows the shape of the distribution and the individual data points.
        - Useful for highlighting clusters, gaps, and outliers.
- Numerical bivariate is data that involves two different numerical variables that have a possible relationship to each other. Displays include scatter plots and line graphs.
    - Scatter Plots

      Good for large data sets, and for data sets in which it is not clear which variable, if any, should be considered the independent variable.

- Line Graphs
  Good for showing trends or cyclical patterns in small or medium-sized data sets in which there is an independent variable and a dependent variable. Often the values of the independent variable are chosen in advance by the person gathering the data. Examples of independent variables may be points in time or treatment amounts and examples of dependent variables might be total sales or average growth.

- Categorical univariate is non-numerical data of only one variable that can be categorized/grouped. Displays include bar charts, line plots, circle graphs, frequency tables and relative frequency tables.
  - Bar Charts (Bar Graphs)
    Good for showing comparisons between categories or between different populations. A bar chart may show frequencies (counts) or relative frequencies (percentages) in each category.
  - Circle Graphs
    Good for illustrating the percentage breakdown of items and visually representing a comparison. Not effective when there are too many categories. Shows how categories represent parts of a whole. A circle graph may show frequencies (counts) or relative frequencies (percentages) in each category.
  - Frequency Tables and Relative Frequency Tables
    This is often the easiest way to display bivariate categorical data. The categories for one variable are listed in the header row of the table and the categories for the other variable are listed in the header column. The frequencies (counts) or relative frequencies (percentages) are listed in the cells for each of the indicated joint categories. Total counts or percentages for the rows may be listed in the final column of the table and total counts or percentages for the columns may be listed in the final row.
  - Segmented Bar Charts
    - Comparison of more than one categorical data sets.
    - Good for showing the composition of the individual parts to the whole and making comparisons.
- Non-numerical data may consist of numbers if the categories are not primarily determined by the numerical size or order of the numbers.
  - For example, the data may answer the question "What is your favorite real number?" and the categories could be "Integers," "Rational numbers that are not integers" and "Irrational numbers."
  - Irrational numbers can be used as identifiers: zip codes, social security numbers, uniform numbers.
- Using the same real-world data (*MTR.7.1*), encourage students to create a variety of data displays appropriate for the data given (*MTR.2.1*). This makes the discussion of the similarities and differences of the displays more robust and allows students to visualize and justify their responses (*MTR.3.1*).
  - This strategy might work best if you present the class with a set of data, group students and ask each group to create a different display using the same data.
  - Each group can then present the strengths and weaknesses of their display as compared to the others (*MTR.5.1*).

    - This should be repeated for each separate data category, see examples above.
- This benchmark references bar charts; however, other benchmarks and the glossary (*Appendix C*) reference bar graph, these terms are used interchangeably without difference.
- Instruction includes student discussions (*MTR.1.1*) regarding the strengths and weaknesses of each data display, and includes the use of appropriate units and labels (*MTR.4.1*).

Common Misconceptions or Errors

- Students may not know how to label displays appropriately or how to choose appropriate units and scaling.
    - For example, they may not know how to create or scale the number line for a line plot, they may confuse frequency and actual data values, or they may not understand that intervals for histograms should be done in equal increments.
- Students may not understand the meaning of quartiles in the box plot.
- Students may not know how to calculate the median with an even number of data values.
- Students may not accurately place data values in increasing order when there are many data points.
- Students may confuse bar charts (for categorical data) and histograms (for numerical data).
- Students may be confused when categorical data consists of numbers that have been categorized in ways that do not primarily reflect the numerical size or order of the numbers. In such cases, it will be helpful to have the student think about whether any of the measures of center (mean, median) or variability (quartiles, range) are meaningful for the data set. If they are, then the data can be considered numerical, because these measures are concerned with the numerical size and order of the data points. If not, then it can be considered categorical.

Instructional Tasks

Instructional Task 1 (MTR.7.1)

The band sold popcorn to raise money. The number of bags each member sold is listed below.

$$\{15, 12, 14, 7, 25, 9, 13, 14, 15, 30, 24, 24, 18, 16, 10, 8, 18, 20, 21, 22\}$$

Part A. Is the data numerical or categorical?

Part B. Is the data univariate or bivariate?

Part C. Which data display would you use to represent this data? Explain your reasoning.

Part D. Write a question that could be used to analyze the number of bags each member sold.

Instructional Task 2 (MTR.2.1, MTR.6.1)

Students were asked if they planned on attending the school concert on Friday. Their responses are listed below.

Key: 1: Yes 2: No 3: Undecided

2 1 1 1 2 1 3 1 2 1 1 2 2 1 1 3 3 1 1 1 2 3 2 1 1

Part A. Is the data numerical or categorical?

Part B. Is the data univariate or bivariate?

Part C. Which data display would you use to represent this data? Explain your reasoning.

Instructional Task 3 (MTR.7.1)

The following data set shows the change in the total amount of students in the incoming freshmen class.

Year	Number of Students in Freshmen Class
1975	345
1980	358
1985	372
1990	320
1995	343
2000	350
2005	350
2010	365

Part A. Is the data numerical or categorical?

Part B. Is the data univariate or bivariate?

Part C. Which variable is the independent variable?

Part D. Choose and create an appropriate data display to represent the information given.

Instructional Items

Instructional Item 1

The bird watching club hikes a different trail each month and records the number of birds they see. The following table shows the types and numbers of birds seen on the most recent hike.

Type of Bird	Number of Birds Seen
Coot	12
Great Egret	5
Green Heron	6
Moorhen	11
Osprey	2
Red-winged Blackbird	20
White Ibis	17
Cattle Egret	27

Which data display would you use to represent this data? Explain your reasoning.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.DP.1.2

Benchmark

MA.912.DP.1.2	Interpret data distributions represented in various ways. State whether the data is numerical or categorical, whether it is univariate or bivariate and interpret the different components and quantities in the display.

Benchmark Clarifications:
Clarification 1: Within the Probability and Statistics course, instruction includes the use of spreadsheets and technology.

Connecting Benchmarks/Horizontal Alignment

- MA.912.DP.2.5, MA.912.DP.2.6
- MA.912.DP.3.1, MA.912.DP.3.2, MA.912.DP.3.3

Terms from the K-12 Glossary

- Bivariate data
- Categorical data
- Numerical data

Vertical Alignment

Previous Benchmarks

Next Benchmarks

- MA.912.FL.4.4
- MA.912.DP.2.2

Purpose and Instructional Strategies

In Algebra I, students interpreted the components of data displays for numerical and categorical data, both univariate and bivariate. In Math for College Liberal Arts, students continue to identify the types of data based on given distributions. In other courses, they will use data displays to compare distributions of data sets to one another and to theoretical distributions.

- For students to have full understanding of numerical/categorical, univariate/bivariate data sets and their displays, instruction should include MA.912.DP.1.1. These benchmarks are not intended to be separated. One is reinforced by the other.
- Numerical univariate includes histograms, stem-and-leaf plots, box plots and line plots.
- Numerical bivariate includes scatter plots and line graphs.
- Categorical univariate includes bar charts, circle graphs, frequency tables and relative frequency tables.
- Categorical bivariate includes segmented bar charts, joint frequency tables and joint relative frequency tables.
- This benchmark reinforces the importance of the use of questioning within instruction.
    - Does this display univariate or bivariate data?
    - Is the data numerical or categorical?
    - What do the different quantities within the data display mean in terms of the context of the situational data?

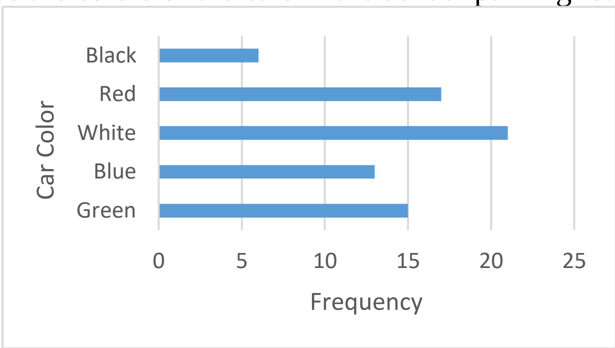
Common Misconceptions or Errors

- Students may not be able to properly distinguish between numerical and categorical data.
- Students may not be able to properly distinguish between univariate and bivariate data.
- Students may not be able to identify and/or interpret the quantities in the various data displays.
- Students may not completely grasp the effect of outliers on the data set; or incorrectly conclude a point is an outlier.
- Students may not be able to distinguish the differences between frequencies and relative frequencies.
- Students may not be able to identify the condition that determines a conditional or relative frequency in a joint table.

Instructional Tasks

Instructional Task 1 (MTR.7.1)

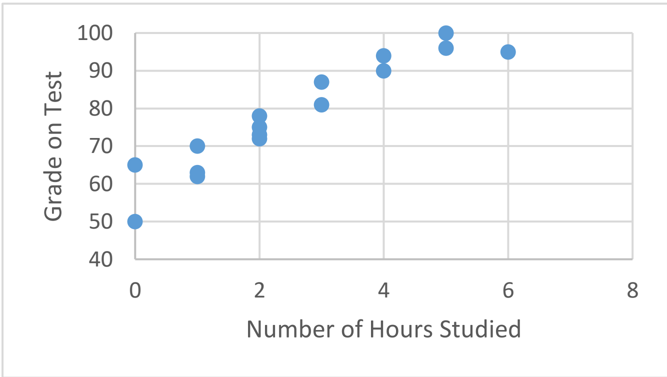
The data below shows the colors of the cars in the school parking lot.



Part A. Does this display univariate or bivariate data?
Part B. Is the data numerical or categorical?
Part C. Which color cars had at least 15 cars on the lot?
Part D. Which color cars had at least 10 and at most 20 cars on the lot?

Instructional Task 2 (MTR.7.1)

The scatter plot shows the number of hours students studied for their science test and their grade on the test.



Part A. Does this display univariate or bivariate data?
Part B. Is the data numerical or categorical?
Part C. What is a possible trend that is shown by the data?

Instructional Items

Instructional Item 1

The chart below shows the square footage of each wrapping paper.

Wrapping Paper	Square Footage on Roll
Stars	75
Snowflakes	40
Candy Canes	55
Presents	75
Snowmen	25

Is the data numerical or categorical? Is it univariate or bivariate?

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.DP.2 *Solve problems involving univariate and bivariate numerical data.*

MA.912.DP.2.1

Benchmark

MA.912.DP.2.1	For two or more sets of numerical univariate data, calculate and compare the appropriate measures of center and measures of variability, accounting for possible effects of outliers. Interpret any notable features of the shape of the data distribution.

Benchmark Clarifications:

Clarification 1: The measure of center is limited to mean and median. The measure of variation is limited to range, interquartile range, and standard deviation.

Clarification 2: Shape features include symmetry or skewness and clustering.

Clarification 3: Within the Probability and Statistics course, instruction includes the use of spreadsheets and technology.

Connecting Benchmarks/Horizontal Alignment

- MA.912.DP.1.1, MA.912.DP.1.2

Terms from the K-12 Glossary

- Interquartile range
- Mean
- Measures of center
- Measures of variability
- Median
- Outlier
- Range

Vertical Alignment

Previous Benchmarks

- MA.912.DP.1.1, MA.912.DP.1.2

Next Benchmarks

- MA.912.DP.2.2

Purpose and Instructional Strategies

In grade 7, students determined an appropriate measure of center or measure of variation to summarize numerical data. In Math for College Liberal Arts, students calculate and compare the measures of center or measures of variation of two or more sets of numerical univariate data. In other classes, students will use technology to calculate and compare two or more sets of numerical data.

- The expectation of this course is to use range and interquartile range as the measures of variation.
- Instruction includes discussion of a measure of center which is a numerical value used to describe the overall clustering of data in a set, or the overall central value of a set of data. Measures of center include the mean and the median.
- Instruction includes finding the mean, or the arithmetic average. The mean is found by adding the data values and dividing by the number of values. The mean is affected by outliers.
  - For example, for the data set $\{3,5,9,8,17,23,25,7\}$, the mean is calculated by dividing the sum of the values, 97, by the number of values, 8: $97/8 = 12.125$.
- Instruction includes finding the median of a set of values. The median is found by finding the middle value of the data when sorted from smallest to largest. The median is not as affected by outliers.
  - For example, for the sorted data set $\{21,23,25,28,27,35,45\}$, the median is the middle value: 28.
- If there is an even number of values, find the mean of the middle two values.
  - For example, for the data set $\{3,5,9,8,17,23,25,7\}$, to find the median the data must first be sorted from smallest to largest: $\{3,5,7,8,9,17,23,25\}$. The middle two values are 8 and 9 so the median is $17/2 = 8.5$.
- Instruction includes measures of variation including the range and the Interquartile Range (IQR). A greater measure of variation shows a greater spread of the data.
- The range is found by finding the difference between the highest and lowest value.
  - For example, given the following 5-number summary: $25, 40, 60, 84, 100$, the range is $100 - 25 = 75$.
- The Interquartile Range (IQR) is the range of the middle 50% of the data and is found by subtracting $IQR = Q_3 - Q_1$.
  - For example, given the following 5-number summary: $25, 40, 60, 84, 100$, $IQR = 84 - 40 = 44$.

Common Misconceptions or Errors

- Students may not put the data values in ascending order when finding the median.

Instructional Tasks

Instructional Task 1 (MTR.5.1, MTR.7.1)

Route 44 and Route 65 are two popular roads in All City, USA. The highway patrol gathered the following data points to show the speeds of cars, in miles per hour, when they were involved in a car accident.

Rt 44	10	25	35	43	72	54	52	25	19	30	35	62	45	59	84	75
Rt 65	35	84	74	45	45	35	30	45	46	49	35	39	45	68	46	65

Part A. Find the 5-number-summary for each road.
Part B. What do you notice about the differences in speeds on the two roads?
Part C. What conclusions might you make about the roads?

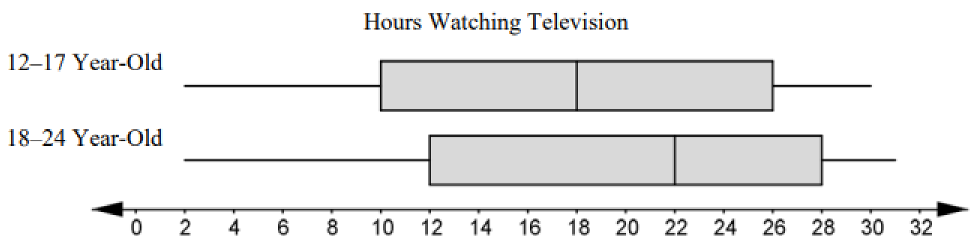
Instructional Task 2 (MTR.4.1, MTR.6.1)

Two companies are posting job openings for entry level workers. Both companies post their average salary as $35,000. Company A has a mean of $35,000 and a median of $17,000. Company B has a mean of $35,000 and a median of $35,000. Which company would be a better company to work for and why?

Instructional Items

Instructional Item 1

Calculate and compare the medians and interquartile ranges for the number of hours per week for the two age groups.



**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.DP.2.4

Benchmark

MA.912.DP.2.4	Fit a linear function to bivariate numerical data that suggests a linear association and interpret the slope and y-intercept of the model. Use the model to solve real-world problems in terms of the context of the data.

Benchmark Clarifications:
Clarification 1: Instruction includes fitting a linear function both informally and formally with the use of technology.
Clarification 2: Problems include making a prediction or extrapolation, inside and outside the range of the data, based on the equation of the line of fit.

Connecting Benchmarks/Horizontal Alignment

- MA.912.DP.1.1, MA.912.DP.1.2

Terms from the K-12 Glossary

- Line of fit
- Numerical data
- Scatter plot

Vertical Alignment

Previous Benchmarks

- MA.912.DP.1.3

Next Benchmarks

Purpose and Instructional Strategies

In Algebra I, students related the slope and y-intercept of a line of fit to association in bivariate numerical data and interpret these features in real-world contexts. In Math for College Liberal Arts, students will continue this work. In other courses, students use the correlation coefficient to measure how well a line fits the data in a scatter plot, and they also work with scatter plots that suggest quadratic and exponential models.

- This is an extension of MA.912.DP.1.1, where students are working with numerical bivariate data (scatter plots and line graphs). It is good to review with students that a scatter plot is a display of numerical data sets between two variables.
  - They are good for showing a relationship or association between two variables.
  - They can reveal trends, shape of trend or strength of relationship trend.
  - They are useful for highlighting outliers and understanding the distribution of data.
  - One variable could be the progression of time, like in a line graph.
- In this benchmark, students are fitting a linear function to numerical bivariate data, interpreting the slope and y-intercept based on the context and using that linear function to make predictions about values that correspond to parts of the graph that lie beyond or within the scatter plot.
- Instruction includes making predictions using the model. Students use interpolation, making predictions within the range of given data, and extrapolation, making predictions outside of the range of given data.
  - For example, if it costs $28 per class at the Gym4You for the first 4 classes that you take, students could write a linear equation for the total cost based on the number of classes taken. Interpolation would consist of predicting the total cost for up to 4 classes at Gym4You. Extrapolation would consist of predicting the total cost for more than 4 classes or less than 0 classes at Gym4You.

Interpolation is often more accurate than extrapolation.
    - Given our example above, a model could predict negative total cost which would not be realistic within the given context.
- During instruction it is important to distinguish the difference between a "line of fit" and the "line of best fit."
    - A "line of fit" is used when students are visually investigating numerical bivariate data that appears to have a linear relationship and can sketch a line (using a writing instrument and straightedge) that appears to "fit" the data. Using this "line of fit" students can estimate its slope and y-intercept and use that information to interpret the context of the data.
    - The "line of best fit" (also referred to as a "trend line") is used when the data is further analyzed using linear regression calculations (the process of minimizing the squared distances from the individual data values to the line), often done with the assistance of technology.

Common Misconceptions or Errors

- Students may not know how to sketch a line of fit.
    - For example, they may always go through the first and last points of data.
- Students may confuse the two variables when interpreting the data as related to the context.
- Students may not know the difference between interpolation (predictions within a data set) and extrapolation (predictions beyond a data set).

Instructional Tasks

Instructional Task 1 MTR.4.1, MTR.5.1, MTR.6.1

A table is given with some data collected. The table shows the average temperature of an afternoon and the number of ice cream cones sold by The Ice Cream Stand in that afternoon.

Average Temperature	Number of Ice Cream Cones Sold
75° F	39
70° F	25
79° F	50
82° F	60
80° F	55
83° F	63
94° F	99
92° F	90
85° F	70

Part A. Create a line of fit based on the data. Compare your line of fit with a partner.
Part B. What is the estimated slope and y-intercept of the line?
Part C. What does the slope mean in terms of the context?
Part D. What does the y-intercept mean in terms of the context? Is this meaningful to the task, why or why not?
Part E. Based on this line, predict the average temperature of the afternoon if The Ice Cream Stand sold 34 ice cream cones that afternoon.

Part F. Based on this line, estimate the number of ice cream cones sold when the temperature is 81°F.

Instructional Items

Instructional Item 1

The data below describes the ages of apple trees at the Happy Apple Orchard and the lengths of their branches, in inches.

Age	2	4	5	8	11	12
Length of Branches (in.)	43	62	71.5	100	128.5	138

Part A. Estimate an equation for a line of fit.
Part B. If an apple tree is 7 years old, estimate the length of its branches.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.DP.2.9

Benchmark

MA.912.DP.2.9	Fit an exponential function to bivariate numerical data that suggests an exponential association. Use the model to solve real-world problems in terms of the context of the data.

Benchmark Clarifications:
Clarification 1: Instruction focuses on determining whether an exponential model is appropriate by taking the logarithm of the dependent variable using spreadsheets and other technology.
Clarification 2: Instruction includes determining whether the transformed scatterplot has an appropriate line of best fit, and interpreting the y-intercept and slope of the line of best fit.
Clarification 3: Problems include making a prediction or extrapolation, inside and outside the range of the data, based on the equation of the line of fit.

Connecting Benchmarks/Horizontal Alignment

- MA.912.AR.5.3, MA.912.AR.5.4, MA.912.AR.5.5, MA.912.AR.5.6
- MA.912.F.1.6, MA.912.F.1.8

Terms from the K-12 Glossary

- Exponential function

Vertical Alignment

Previous Benchmarks

- MA.912.F.1.5, MA.912.F.1.6, MA.912.F.1.8
- MA.912.AR.5.4, MA.912.AR.5.6

Next Benchmarks

Purpose and Instructional Strategies

In Algebra I, students wrote exponential functions that modeled relationships characterized by having a constant percent of change per unit interval and found a line of fit for linear bivariate numerical data. In Math for College Liberal Arts, students recognize exponential functions by the constant percent rate of change per unit interval and fit exponential functions to appropriate bivariate numerical data. In other courses, students will use the logarithm to determine whether an exponential model is appropriate.

- For students to have full understanding of exponential functions, instruction includes MA.912.AR.5.3, MA.912.AR.5.4, MA.912.AR.5.5 and MA.912.AR.5.6.
- Growth or decay of a function can be defined as a key feature (constant percent rate of change) of an exponential function and useful in understanding the relationships between two quantities.
- Instruction includes reviewing the exponential form $f(x) = a(1 + r)^x$ where a is the initial amount, r is the growth rate, and x is time.
- Instruction includes making predictions using the model. Students use interpolation, making predictions within the range of given data, and extrapolation, making predictions outside of the range of given data.
    - For example, if students were collecting data on growth of humans between ages 0 to 5 on a scatter plot and wrote an exponential function representing the data. Interpolation would be predicting heights between the ages of 0 to 5. If students wanted to predict the heights of age 14, this would be extrapolation.
- Interpolation is often more accurate than extrapolation.
    - Given our example above, a model could predict a height of 15 feet which would not be realistic within the given context.

Common Misconceptions or Errors

- Students may confuse when an exponential function is needed, rather than a linear or quadratic function, given a table or written description.
- Students may not know the difference between interpolation (predictions within a data set) and extrapolation (predictions beyond a data set).

Instructional Tasks

Instructional Task 1 (MTR.5.1, MTR.7.1)

Joshua is saving money in a savings account where interest is compounded yearly. The chart below shows how much is in Joshua's account after a number of years.

Years	1	3	5	7	9
Amount	\$1325	\$1488.80	\$1672.80	\$1879.50	\$2111.80

Part A. What ratio describes how much Joshua's amount increases each year?
Part B. What does this ratio describe in this situation?
Part C. How much did Joshua originally deposit in his account?
Part D. Write an exponential function that describes the amount of money in his account after n years.
Part E. How much would be in his account after 10 years? 15 years?
Part F. How would the function change if Joshua originally deposited \$15,000? How much would he have in his account after 10 years?

Instructional Items

Instructional Item 1

In the early 2000s the exponential equation $y = 1.4457(1.0137)^x$ was used to model the population of the world where y represents the population, in billions of people, and x represents the years since 1900.

Part A. Based on the given equation, estimate the population in the year 2020.

Part B. Is this model still representative of the population today?

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.DP.4 *Use and interpret independence and probability.*

MA.912.DP.4.1

Benchmark

MA.912.DP.4.1	Describe events as subsets of a sample space using characteristics, or categories, of the outcomes, or as unions, intersections or complements of other events.

Connecting Benchmarks/Horizontal Alignment

- MA.912.LT.5.1, MA.912.LT.5.4, MA.912.LT.5.5

Terms from the K-12 Glossary

- Event
- Sample space

Vertical Alignment

Previous Benchmarks

- MA.7.DP.2.1
- MA.8.DP.2.1, MA.8.DP.2.2, MA.8.DP.2.3

Next Benchmarks

Purpose and Instructional Strategies

In middle grades, students began to identify sample spaces and explored basic probability by using theoretical probability for repeated experiments. In Mathematics for College Liberal Arts, students break the sample space up into subsets. Students also identify the union, intersection, and complement of other events.

- Within this benchmark, students should use their knowledge of Venn Diagrams and set notation from the Logic and Discrete Theory (LT) benchmarks within their work of these Data Analysis and Probability (DP) benchmarks. The expectation is students are working with multiple events (2 or more).
- Instruction includes finding the sample space for an event and then subsets of that sample space.
    - For example, the sample space for rolling a die is $S = \{1, 2, 3, 4, 5, 6\}$. The sample space for rolling an even number, which is a subset, is $S = \{2, 4, 6\}$.
- Instruction includes the use of diagrams in order to explore and illustrate concepts of complements, unions, intersections, differences and products of two sets.

- The intersection of two sets is denoted as $A \cap B$ and consists of all elements that are both in A and B.
- The complement of a set can be denoted as A^c, $\bar{A}$, A' or $\sim A$. This is the set of all elements that are in the universal set S but not in A. Complements can be of unions or intersections.

- Instruction includes finding sample spaces for multiple events.
  - For example, find the sample space of rolling an even number on a six-sided die and flipping a head on a coin.
    - Sample space for rolling a six-sided die: $R = \{1, 2, 3, 4, 5, 6\}$.
    - Sample space for rolling an even number on a six-sided die: $E = \{2, 4, 6\}$.
    - Sample space for flipping a coin: $F = \{H, T\}$.
    - Sample space for flipping a head: $H = \{H\}$.
    - Sample space for rolling a die and flipping a coin: $S = \{1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T, 6H, 6T\}$.
    - Sample space for rolling an even number and flipping a head: $A = \{2H, 4H, 6H\}$.
    - Sample space for not rolling an even number and not flipping a head: $B = \{1T, 3T, 5T\}$.
- Instruction includes finding the sample space of union and intersections.
  - For example, a deck of 16 cards has 4 red hearts, 4 red diamonds, 4 black spades, and 4 black clubs, numbered 1 to 4.
    - Sample Space for the deck of cards: $S = \{1H, 2H, 3H, 4H, 1D, 2D, 3D, 4D, 1S, 2S, 3S, 4S, 1C, 2C, 3C, 4C\}$.
    - Sample Space for the intersection of red and 3s:
      - Define the subset of Red: $R = \{1H, 2H, 3H, 4H, 1D, 2D, 3D, 4D\}$.
      - Define the subset of 3s: $3s = \{3H, 3D, 3S, 3C\}$.
      - Define the intersection of Red and 3s: $R \cap 3s = \{3H, 3D\}$.
    - Sample Space for the union of clubs and not diamonds:
      - Define the subset of clubs: $C = \{1C, 2C, 3C, 4C\}$.
      - Define the subset of not diamonds: $D' = \{1H, 2H, 3H, 4H, 1S, 2S, 3S, 4S, 1C, 2C, 3C, 4C\}$.
      - Define the union of clubs and not diamonds: $C \cup D' = \{1H, 2H, 3H, 4H, 1S, 2S, 3S, 4S, 1C, 2C, 3C, 4C\}$.

Common Misconceptions or Errors

- If finding the unions when the events are not mutually exclusive, students often count the intersection more than once because it is included in multiple sets.
- Students may forget about the values that are not in the sets.
- Students may confuse the symbols used in basic set notation.

Instructional Tasks

Instructional Task 1 (MTR.7.1)

The table below provides information on 10 students within Student Government. For each student, their sex, age, whether or not they plan to go to college, if they play a sport, and how many honors courses they are currently enrolled in.

Position in Student Government	Sex	Age	Plan to Go to College	Play a Sport?	Number of Honors Courses
President	Male	18	Yes	No	3
Vice President	Female	18	No	Yes	4
Secretary	Female	17	Yes	No	2
Treasurer	Male	18	No	Yes	2
Parliamentarian	Female	18	Yes	No	5
Historian	Male	17	Yes	No	7
Ad Hoc 1	Male	17	Yes	Yes	3
Ad Hoc 2	Male	18	Yes	Yes	5
Ad Hoc 3	Female	16	No	Yes	4
Ad Hoc 4	Male	16	No	No	2

Part A. What is the sample space for number of honors courses they are enrolled in currently?

Part B. What outcomes from the sample space in Part A are in the event that the student plans to go to college?

Part C. Consider the following 3 events, for each list which students, by their positions in student government, would be make up the event.

- F = The selected student is female
- S = The selected student plays a sport
- A = The selected student is less than 18 years old

Part D. Based on part C, which outcomes are in the following events? Describe the characteristics of each sample space.

- $F \cup S$
- $(F \cup S)'$
- $F \cap A$
- A^C

Instructional Items

Instructional Item 1

What is the sample space of not rolling a number greater than 4 on a six-sided die?

Instructional Item 2

What is the sample space of not rolling a number greater than 4 on a six-sided die and flipping a head on a coin?

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.DP.4.2

Benchmark

MA.912.DP.4.2	Determine if events A and B are independent by calculating the product of their probabilities.

Connecting Benchmarks/Horizontal Alignment

Terms from the K-12 Glossary

- Event

Vertical Alignment

Previous Benchmarks

- MA.7.DP.2.2, DP.2.3, DP.2.4
- MA.8.DP.2.2, DP.2.3

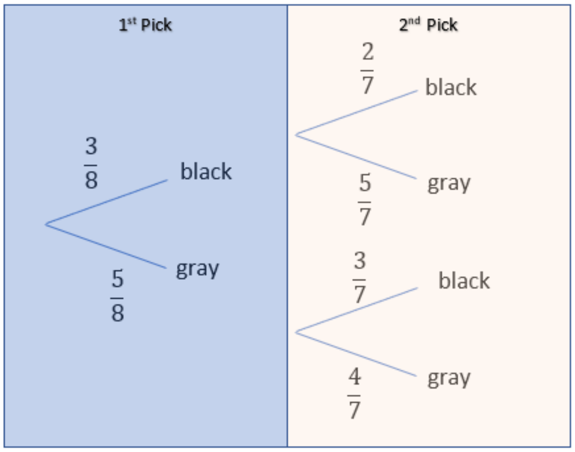
Next Benchmarks

Purpose and Instructional Strategies

In middle grades, students began working with theoretical probabilities and comparing them to experimental probability. In Mathematics for College Liberal Arts, students determine if two events are independent of each other.

- Independence means the outcome of one event does not influence the outcome of the second event.
    - For example, a pair of independent events are if you are rolling a die and then flipping a coin. The number on the die has no effect on whether the coin will land on heads or tails. Therefore, these events are considered to be independent.
    - Another example would be if students draw two cards from a deck and replaced them each time from the deck, these events would be considered to be independent.
- Dependences means that the outcome of one event influences the outcome of the second.
    - For example, if you are drawing two cards from a deck and you do not replace them each time from the deck, these events would be considered to be dependent.
- For this benchmark, students determine independence if the probability of Event A and B is equivalent to the probability of A times the probability of B. Students can use see if events are independent by determining if $P(A \cap B) = P(A) \times P(B)$.
    - For example, if we roll a six-sided die and then flip a coin. These are considered to be independent events.
      The probability of rolling a 1 is $\frac{1}{6}$.
      The probability of a coin landing on heads is $\frac{1}{2}$. The probability of rolling a one and getting a coin landing on a head is $\frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$.
      Students can list the outcomes
      $\{(1,H),(2,H),(3,H),(4,H),(5,H),(6,H),(1,T),(2,T),(3,T),(4,T),(5,T),(6,T)\}$.
      There are 12 outcomes equally likely to occur, therefore the probability of rolling a 1 and a landing on heads is $\frac{1}{12}$.

- For example, if we were drawing socks from our sock drawer and we were not replacing them, this would be considered dependent events. Susie's sock drawer had 3 pairs of black socks and 5 pairs of gray socks.



So the probabilities of the different color combinations are:
picking two black socks: $\frac{3}{8} \times \frac{2}{7} = \frac{6}{56}$
picking one black sock and one gray sock: $\left(\frac{3}{8} \times \frac{5}{7}\right) + \left(\frac{5}{8} \times \frac{3}{7}\right) = \frac{30}{56}$
picking two gray socks: $\frac{5}{8} \times \frac{4}{7} = \frac{20}{56}$
Since the probability of the second pick depends on which sock you choose first, these are considered to be dependent events.

- Be sure to distinguish independence from mutually exclusive events. Mutually exclusive events are events that cannot occur simultaneously. This is noted as $P(A \cap B) = 0$.
  - Note that $P(A \cap B)$ is the same as $P(B \cap A)$.

Common Misconceptions or Errors

- Students may confuse what it means to be dependent and independent.
- Students may confuse independence with mutually exclusive events.
- Students may struggle to convert fractions, decimals, and percentages.

Instructional Tasks

Instructional Task 1 (MTR.6.1)

One card is elected at random from a deck of 6 cards. Each card has a number and either a spade or a club: {3♣, 5♣, 2♠, 9♣, 9♠, 7♣}.

Part A. Let C be the event that the selected card is a club, and F be the event that the selected card is a 5. Are the events C and F independent? Justify your answer with calculation.

Part B. Let S be the event that the selected card is a spade, and N be the event that the selected card is a 9. Are the events S and N independent? Justify your answer with calculation.

Instructional Items

Instructional Item 1

Probabilities for events A, B and C are described below.

$$P(A) = 0.20$$
$$P(B) = 0.55$$
$$P(C) = 0.36$$
$$P(A \cap B) = 0.110$$
$$P(A \cap C) = 0.560$$
$$P(B \cap C) = 0.198$$

Part A. Are the events A and B independent?
Part B. Are the events A and C independent?
Part C. Are the events B and C independent?

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.DP.4.3

Benchmark

MA.912.DP.4.3	Calculate the conditional probability of two events and interpret the result in terms of its context.

Connecting Benchmarks/Horizontal Alignment

Terms from the K-12 Glossary

- Event
- Sample space

Vertical Alignment

Previous Benchmarks

- MA.7.DP.2.2, MA.7.DP.2.3, MA.7.DP.2.4
- MA.8.DP.2.2, MA.8.DP.2.3
- MA.912.DP.3.1

Next Benchmarks

Purpose and Instructional Strategies

In middle grades, students began working with theoretical probabilities and comparing them to experimental probability. In Algebra I students determined frequencies from two-way tables. In Math for College Liberal Arts, students calculate the conditional probability of two events and interpret the results in context of the problem.

- Conditional probability of an event is the probability that the event will occur given the knowledge that another event has already occurred.
- Conditional probability is written as $P(B|A)$ and is read as the probability of event B given event A.
- Instruction includes understanding that the probability of event A given event B is different than the probability of event B given event A. Students should note that the formulas for both events are different.

  The conditional probability of $P(B|A) = \frac{P(A \cap B)}{P(A)}$.

- For example, Jackson is rolling a fair six-sided die. He wants to find the probability that the number rolled is a five, given that it is odd.
  The sample space for this experiment is the set $S = \{1, 2, 3, 4, 5, 6\}$ consisting of six equally likely outcomes.
  F denotes the event "a five is rolled" and O denotes the event "an odd number is rolled."

$$F = \{5\} \text{ and } O = \{1, 3, 5\}$$
$$P(F|O) = \frac{P(F \cap O)}{P(O)}$$
$$F \cap O = \{5\} \cap \{1,3,5\}; \text{ so } F \cap O = \{5\}$$
$$P(F \cap O) = \frac{1}{6}$$
$$\text{Since } O = \{1, 3, 5\},\ P(O) = \frac{3}{6} \text{ or } \frac{1}{2}$$
$$P(F|O) = \frac{P(F \cap O)}{P(O)} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3}$$

- Informally, conditional probability of $P(B|A)$ can be thought of as restricting the sample space to only include the outcomes in event A.
  - For example, Jackson is rolling a fair six-sided die. He wants to find the probability that the number rolled is a five, given that it is odd.
    We can reduce our sample space to the odd numbers only.

$$O = \{1, 3, 5\}$$

One of these outcomes is a 5, therefore $P(rolling\ a\ 5|Odd) = \frac{1}{3}$.

The conditional probability of $P(A|B\ ) = \frac{P(A \cap B)}{P(B)}$.

- For example, Jackson is rolling a fair six-sided die. He wants to find the probability that the number rolled is an odd, given that it is five.
  The sample size for this experiment is the set $S = 1, 2, 3, 4, 5, 6$ consisting of six equally likely outcomes.
  F denotes the event "a five is rolled" and O denotes the event "an odd number is rolled."

$$F = 5 \text{ and } O = 1, 3, 5$$
$$P(O|F) = \frac{P(O \cap F)}{P(F)}$$
$$O \cap F = 1, 3, 5 \cap\ 5; \text{ so } O \cap F = 5$$
$$P(O \cap F) = \frac{1}{6}$$
$$\text{Since } F = 5,\ P(F) = \frac{1}{6}$$
$$P(O|F) = \frac{P(O \cap F)}{P(F)} = \frac{\frac{1}{6}}{\frac{1}{6}} = 1$$

- Informally, conditional probability of $P(A|B)$ can be thought of as restricting the sample space to only include the outcomes in event B.
  - For example, Jackson is rolling a fair six-sided die. He wants to find the

probability that the number rolled is a odd, given that it is five.
We can reduce our sample space to the number 5.
This one outcome is odd, therefore $P(odd|rolling\ a\ 5) = \frac{1}{1} = 1$.

- Conditional probabilities can be observed in tree diagrams as the branch stemming from another branch which may be a helpful visual for some students to understand the meaning of conditional probabilities.
- Instruction includes the use of two way tables.
- It is not the expectation for this benchmark for students to memorize formulas.

Common Misconceptions or Errors

- Students may think the symbol used for conditional probability is a slash that would be used to represent division and simply divide the probability of A by the probability of B.
- Students may get confused as to which event's probability should be the denominator.
- Students may get confused when working with a two-way table that they need to restrict their answer to a certain section of the table that is defined by the "given" conditional piece.
  - For example, when given the condition of male they are only looking in the row or column containing males to get the restricted sample space.

Instructional Tasks

Instructional Task 1

All the students at All Florida High School were surveyed and they were classified according to year and whether or not they have one or both ears pierced. One student is randomly selected.

	Pierced	**Not Pierced**	Total
Freshman	183	182	365
Sophomore	177	118	295
Junior	139	66	205
Senior	101	34	135
Total	600	400	1000

Part A. Are the events of having your ears pierced and the year you are in school independent or dependent? How do you know?
Part B. What is the probability of the selected student having pierced ears?
Part C. What is the probability of the selected student having pierced ears given that the student is a sophomore?
Part D. What is the probability of the selected student having pierced ears given that the student is not a sophomore?
Part E. What is the probability of the selected student being a sophomore given they have pierced ears?

Instructional Items

Instructional Item 1

Compute the following probabilities in connection with the roll of a single fair six-sided die.

Part A. The probability that the roll is even.
Part B. The probability that the roll is even, given that it is not a four.
Part C. The probability that the roll is even, given that it is not a three.

Instructional Item 2

A special deck of 16 cards has 4 that are green, 4 yellow, 4 blue, and 4 red. The four cards of each color are numbered from one to four. A single card is drawn at random. Find the following probabilities.

Part A. The probability that the card drawn is red.
Part B. The probability that the card drawn is red, given that it is not blue.
Part C. The probability that the card drawn is red, given that it is not a four.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.DP.4.4

Benchmark

MA.912.DP.4.4 Interpret the independence of two events using conditional probability.

Connecting Benchmarks/Horizontal Alignment

Terms from the K-12 Glossary

- Conditional relative frequency
- Event
- Experimental probability
- Frequency table
- Joint frequency
- Sample space
- Theoretical probability

Vertical Alignment

Previous Benchmarks

- MA.7.DP.2.2, MA.7.DP.2.3, MA.7.DP.2.4
- MA.8.DP.2.2, MA.8.DP.2.3

Next Benchmarks

Purpose and Instructional Strategies

In middle grades, students began working with theoretical probabilities and comparing them to experimental probability. In Mathematics for College Liberal Arts, students determine if two events are independent of each other.

- Independence in this sense means that knowing whether one event occurred does not change the probability of the other event occurring. For this benchmark, students determine independence using conditional probabilities. Two events, A and B, are independent if $P(A \mid B) = P(A)$ and $P(B \mid A) = P(B)$.

- Students also use the product of probabilities to check independence (MA.912.DP.4.2). If $P(A \cap B) = P(A) \times P(B)$, then the events are independent.
- Be sure to distinguish independence from mutually exclusive events.
  - In mutually exclusive events $(A \cap B) = 0$. This means that the two events cannot occur at the same time.
- Instruction includes the understanding that $P(A \mid B) \neq P(B \mid A)$ unless $P(A) = P(B)$.
- When we check for independence in real-world data sets, it's rare to get perfectly equal probabilities. We often assume that events are independent and test that assumption on sample data. If the probabilities are significantly different, then we conclude the events are not independent.

Common Misconceptions or Errors

- Students may confuse what it means to be dependent and independent.
- Students may confuse independence with mutually exclusive events.
- Students may struggle to convert fractions, decimals, and percentages.
- Students may think the symbol used for conditional probability is a slash that would be used to represent division and simply divide the probability of A by the probability of B.
- Students may get confused as to which event probability should be the denominator.
- Students may get confused when working with a two-way table that they need to restrict their answer to a certain section that is from the "given" conditional piece.
  - For example, when given the condition of male they are only looking in the row or column containing males to get the total.
- Students who struggle to decode the terminology and notation may struggle to understand what is being asked by the questions.

Instructional Tasks

Instructional Task 1

The table below displays the results of a survey of eating preferences.

	Vegetarian	**Not a Vegetarian**	Total
Male	14	36	50
Female	15	35	50
Total	29	71	100

Part A. Name two events that can be represented by the two-way table.

Part B. Are the two events in Part A independent events? Explain why or why not using conditional probability.

Instructional Items

Instructional Item 1

There are 16 cards in a deck of cards. There are 4 red cards, 4 green cards, 4 blue cards and 4 yellow cards. The cards in each color are numbered 1 to 4. Are picking the number 4 and picking a yellow card independent events? Why or why not?

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.DP.4.5

Benchmark

MA.912.DP.4.5 Given a two-way table containing data from a population, interpret the joint and marginal relative frequencies as empirical probabilities and the conditional relative frequencies as empirical conditional probabilities. Use those probabilities to determine whether characteristics in the population are approximately independent.

Example: A company has a commercial for their new grill. A population of people are surveyed to determine whether or not they have seen the commercial and whether or not they have purchased the product. Using this data, calculate the empirical conditional probabilities that a person who has seen the commercial did or did not purchase the grill.

Benchmark Clarifications:
Clarification 1: Instruction includes the connection between mathematical probability and applied statistics.

Connecting Benchmarks/Horizontal Alignment

- MA.912.DP.1.1, MA.912.DP.1.2

Terms from the K-12 Glossary

- Conditional relative Frequency
- Joint frequency
- Joint relative frequency
- Population

Vertical Alignment

Previous Benchmarks

- MA.912.DP.3.1

Next Benchmarks

Purpose and Instructional Strategies

In Algebra I, students studied bivariate categorical data, displayed the data in tables showing joint and marginal frequencies, and discussed possible associations. This was extended to conditional relative frequencies in Algebra I Honors. In Math for College Liberal Arts, students calculate empirical probabilities and conditional empirical probabilities and use these calculations to determine and discuss independence.

- Instruction focuses on real-world data and interpreting the joint and marginal relative frequencies as empirical probabilities, or experimental probabilities, meaning they are based on existing, collected data. Additionally, the conditional relative frequencies will be interpreted in context as empirical conditional probabilities.
- Instruction includes discussing the differences between joint, marginal and conditional relative frequencies.
  - Marginal relative frequencies
    Guide students to understand that the total column and total row are in the "margins" of the table, thus they are referred to as the marginal relative frequencies.
  - Joint relative frequencies
    Guide students to connect that the word joint refers to the coming together of

more than one, therefore the term joint relative frequency refers to combination of two categories or conditions happening together.

- Conditional relative frequencies
  Guide students to connect that the word conditional refers to imposing a constraint on one of the two variables, therefore the term conditional relative frequency refers to the percentage in a category for one variable when one has put a constraint on the other variable.

- Depending on a student's pathway, they may not have had experience with relative frequency tables. Instruction includes making the connection between frequency and relative tables to find empirical probabilities and empirical conditional probabilities.
  - For example, the relative frequency table below describes whether a student completed a test review sheet and whether they passed the test.

	Passed Test	**Did Not Pass Test**	Total
Completed Test Review	54%	6%	60%
Did Not Complete Test Review	20%	20%	40%
Total	74%	26%	100%

    - To determine the marginal relative frequency of those who completed the test review, one would look to the percentage of those who completed the test review, or 60%. One could say that the empirical probability of a student completing the test review is 60%.
    - To determine the joint relative frequency of those who passed the test and completed the test review, one would look to the intersection of those categories within the table, or 54%. One could say that the empirical probability that a student passed the test and completed the test review is 54%.
    - To determine the conditional relative frequency of those who passed the test given that they completed the review, one could take the ratio between the joint relative frequency for passing the test in the first row and the marginal relative frequency in the first row, which is $\frac{54\%}{60\%}$ which is equivalent to 90%. One could say that the empirical conditional probability that a student passed the test given that they completed the review is 90%.

- Instruction includes the connection between mathematical (or theoretical) probability and applied statistics. While mathematical probability deals with predicting the likelihood of an event yet to happen, applied statistics analyzes the frequency of a past event in order to make conclusions about a population. Essentially, we use probabilities to draw conclusions from given data.
  - One conclusion to be analyzed from these probabilities is whether characteristics in the population are approximately independent, as seen in MA.912.DP.4.4 and MA.912.DP.4.6.
  - Students may use their prior knowledge of the word independent to mean not relying on another, and believe that conditional probabilities should be different when events are independent. Emphasize that having equivalent conditional probabilities means the probability of an event is the same no matter whether the other event occurs.

Common Misconceptions or Errors

- Students may not be able to distinguish the differences between marginal, joint and conditional relative frequencies.
- Students may not be able to distinguish the differences between frequencies and relative frequencies.

Instructional Tasks

Instructional Task 1 (MTR.2.1, MTR.4.1, MTR.7.1)

The United States Census Bureau collected data between 2008 and 2012 on the number of workers in various cities who commuted to work by bicycle and those who walked. In looking at data for both Tampa, Florida, and Miami, Florida, there are 14,380 workers who either walk or use a bicycle to commute to work. About 47.43% of this total are people who walk to work in Miami, with about 56.04% of the people being workers in Miami. About 33.66% of those working in Tampa commute to work using a bicycle.

Part A. Construct a relative frequency table with the given information.

	Miami	**Tampa**	Total
Walk	47.43%		
Bike			
Total			100%

Part B. Determine and interpret at least two empirical probabilities from the table in Part A. Compare your probabilities with a partner.

Part C. Determine and interpret at least two empirical conditional probabilities from the table in Part A. Compare your probabilities with a partner.

Part D. Are the events commuting to work via bicycle and working in Tampa independent? Explain.

Instructional Items

Instructional Item 1

The table displays the results of a survey of eating preferences of a sample of high school students. Use the data in the two-way frequency table below to answer the following questions.

	Vegetarian	**Not a Vegetarian**	Total
Male	12%	38%	50%
Female	17%	33%	50%
Total	29%	71%	100%

Part A. In the context of the data, interpret the empirical conditional probability of $\frac{17}{50}$.

Part B. In the context of the data, interpret the empirical conditional probability of $\frac{17}{29}$.

Instructional Item 2

The table below represents dropout rates in 1970 for children between 16 and 24 years old. Are sex and dropping out of high school approximately independent?

	High School Dropout	**Not a High School Dropout**	Total
Male	7%	43%	50%
Female	8%	42%	50%
Total	15%	85%	100%

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.DP.4.6

Benchmark

MA.912.DP.4.6 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations.

Connecting Benchmarks/Horizontal Alignment

Terms from the K-12 Glossary

Vertical Alignment

Previous Benchmarks

- MA.7.DP.2.2, MA.7.DP.2.3, MA.7.DP.2.4
- MA.8.DP.2.2, MA.8.DP.2.3
- MA.912.DP.3.1

Next Benchmarks

- MA.912.DP.3.2, MA.912.DP.3.3, MA.912.DP.3.5

Purpose and Instructional Strategies

In Algebra I, students studied associations in bivariate categorical data using conditional relative frequencies. In Math for College Liberal Arts, students work in more depth with conditional probabilities and independence in real-world contexts.

- Instruction includes tasks asking students to explain the meaning of independence in a simple context, as well as what it would mean for two events to not be independent. Students should analyze and think critically about relationships between two events that may or may not appear to be related.
- Provide common examples of independent and dependent events and ask students to provide examples of their own for both cases.
  - Independent: Lisa ate breakfast, and she went to school.
  - Dependent: Jim loaded his videogame disk, and started playing his videogame.
- Remind students that given data, independence can be calculated, or verified. In some cases, situations that intuitively seem independent, may be correlated. Emphasize that this correlation does not imply cause or dependence, but rather only shows that the two events are not independent. (Refer to MA.912.DP.1.3 from Algebra I.) Two-way tables can be used to assist with this discussion.
  - For example, ask students to answer a series of questions based on given data. The table below shows caffeine preference for mathematics and engineering students based on a survey of 200 college students.

	Daily Caffeine	
Major	Coffee	Energy Drink
Mathematics	24	76
Engineering	53	47

Students can describe two conditional probabilities in everyday language that can be determined from the "Mathematics" row in the table. Everyday language could include the probability of choosing a student who prefers coffee given they are math major; or of those who are math majors choosing an energy drinker. Students can describe two conditional probabilities in everyday language that can be determined from the "Coffee" column in the table. Everyday language could include the probability of choosing a student who is a math major given they drink coffee; or the probability of choosing an engineering student from the coffee drinkers.

Students can determine whether students with a mathematics major more likely to drink coffee or if students with an engineering major are. To do so, students would need to determine if the events are independent (MA.912.DP.4.4).

- Students use their prior knowledge of the word independent to mean not relying on another, and believe that conditional probabilities should be different when events are independent. Emphasize that having equivalent conditional probabilities means the probability of an event is the same no matter whether the other event occurs.

Common Misconceptions or Errors

- Students misinterpret the word independence in relation to probability.
- Students may believe equal conditional probabilities means two events depend on each other in order to be the same. To address this misconception, discuss the meaning in context to emphasize the probability of one event is just as likely whether the other event occurs or not.
- Students may assume independence based on the description of the events alone.

Instructional Tasks

Instructional Task 1 (MTR.7.1)

One hundred people were surveyed and asked their preference for watching movies via streaming services or a movie theater. The results are shown in the table below.

	Streaming Service	**Movie Theater**	Total
Children (under 18 years old)	37	21	58
Adults	8	34	42
Total	45	55	100

Part A. Describe three possible conditional probabilities using everyday language given this table.

Part B. Find the probability that a person surveyed prefers streaming.

Part C. Find the probability that a person surveyed prefers streaming given they are a child.

Part D. Are the events of prefers streaming and being a child independent? Explain.

Instructional Items

Instructional Item 1

Part A. Today there is a 55% chance of rain, a 20% chance of lightning, and a 15% chance of lightning and rain together. Are the two events “rain today” and “lightning today” independent events? Justify your answer.

Part B. Now suppose that today there is a 60% chance of rain, a 15% chance of lightning, and a 20% chance of lightning if it’s raining. What is the chance of both rain and lightning today?

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.DP.4.7

Benchmark

MA.912.DP.4.7 Apply the addition rule for probability, taking into consideration whether the events are mutually exclusive, and interpret the result in terms of the model and its context.

Connecting Benchmarks/Horizontal Alignment

- MA.912.LT.5.4, MA.912.LT.5.5

Terms from the K-12 Glossary

- Event

Vertical Alignment

Previous Benchmarks

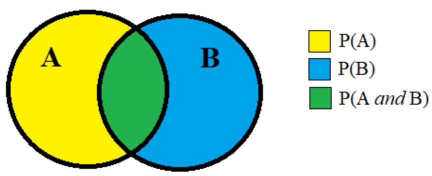
- MA.7.DP.2.2, MA.7.DP.2.3, MA.7.DP.2.4
- MA.8.DP.2.2, MA.8.DP.2.3

Next Benchmarks

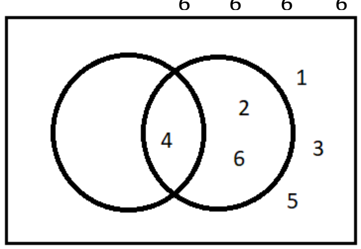
Purpose and Instructional Strategies

In middle grades, students found and compared experimental and theoretical probabilities for both single and repeated experiments. In Math for College Liberal Arts, students expand on their work with probability by applying the addition rule for probability and interpreting those results in context.

- Instruction includes connecting set operations (MA.912.LT.5.4) and exploring relationships using Venn Diagrams (MA.912.LT.5.5) when working to define the addition rule for probability: $P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B)$.



  - This can also be described as $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
  - For example, when rolling a die, the probability for rolling a 4 or an even number can be calculated using the additional rule for probability, and expressed visually with Venn Diagrams.
    - $P(roll\ 4) = \frac{1}{6}$, $P(roll\ even) = \frac{3}{6}$, and $P(roll\ 4\ and\ even) = \frac{1}{6}$
      So, $P(roll\ 4\ or\ even) = \frac{1}{6} + \frac{3}{6} - \frac{1}{6} = \frac{3}{6}$ or $\frac{1}{2}$.



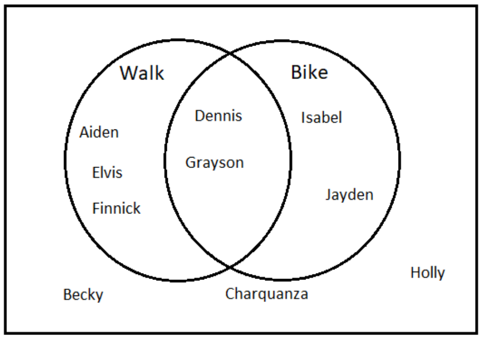
- Emphasize that the additional rule for probability considers the probability or A or B to mean either A or B or both.
- When developing the additional rule for probability, emphasize use of examples where the elements in each set may be easily counted. This will allow students to see that the

elements in the intersection are counted twice, and therefore must be subtracted out once to avoid this double counting.

- For example, 10 students were surveyed about how they get to school each day. Aiden walks, Becky takes the bus, Charquanza takes the bus, Dennis rides a bike or walks, Elvis walks, Finnick walks, Grayson rides a bike or walks, Holly rides the bus, Isabel rides a bike, and Jayden rides a bike. If a student is chosen at random from this group, find the probability that they either ride a bike or walk to school.

$$P(bike\ or\ walk) = P(bike) + P(walk) - P(bike\ and\ walk)$$
$$P(bike\ or\ walk) = \frac{4}{10} + \frac{5}{10} - \frac{2}{10}$$
$$P(bike\ or\ walk) = \frac{7}{10}$$

- Visually, we can show how Dennis and Grayson are double counted using a Venn Diagram, and therefore $\frac{2}{10}$ must be subtracted.



- Instruction includes leading students to a definition of mutually exclusive by presenting cases where the probability of two events occurring simultaneously is 0, or impossible. For example, when rolling a die, the events rolling a 1 and rolling an even number are mutually exclusive.

$$P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B)$$
$$P(1\ or\ even) = P(1) + P(even) - P(1\ and\ even)$$
$$P(1\ or\ even) = \frac{1}{6} + \frac{3}{6} - 0$$
$$P(1\ or\ even) = \frac{4}{6} \text{ or } \frac{2}{3}$$

- In this case, as there is no intersection, meaning $P(A\ and\ B) = 0$, we can use the formula $P(A\ or\ B) = P(A) + P(B)$.

Common Misconceptions or Errors

- Students may not understand mutually exclusive events.
- Students may neglect to subtract the intersection when using the addition rule for probability.

Instructional Tasks

Instructional Task 1 (MTR.4.1)

Today there is a 55% chance of rain, a 20% chance of lightning, and a 15% chance of lightning and rain together.

Part A. Are the two events "rain today" and "lightning today" mutually exclusive? Justify your answer.

Part B. What is the chance that we will have rain or lightning today?

Part C. Now suppose that today there is a 50% chance of rain, a 60% chance of rain or lightning, and a 15% chance of rain and lightning. What is the chance that we will have lightning today?

Instructional Items

Instructional Item 1

At Mom's diner, everyone drinks coffee. Let C represent the event that a randomly-selected customer puts cream in their coffee. Let S represent the event that a randomly-selected customer puts sugar in their coffee. Suppose that after years of collecting data, Mom has estimated the following probabilities:

$$P(C) = 0.6P(S)$$
$$= 0.5P(C \text{ or } S)$$
$$= 0.7$$

Estimate $P(C \text{ and } S)$ and interpret this value in the context of the problem.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.DP.4.8

Benchmark

MA.912.DP.4.8	Apply the general multiplication rule for probability, taking into consideration whether the events are independent, and interpret the result in terms of the context.

Connecting Benchmarks/Horizontal Alignment

Terms from the K-12 Glossary

- Event

Vertical Alignment

Previous Benchmarks

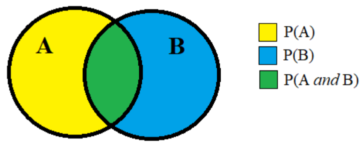
- MA.7.DP.2.2, MA.7.DP.2.3, MA.7.DP.2.4
- MA.8.DP.2.2, MA.8.DP.2.3

Next Benchmarks

Purpose and Instructional Strategies

In middle grades, students found and compared experimental and theoretical probabilities for both single and repeated experiments. In Math for College Liberal Arts, students expand on their work with probability by applying the general multiplication rule for probability and interpreting those results in context.

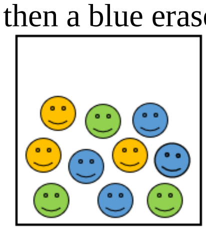
- While the addition rule for probability deals with the probability that either one event or another event will occur, the multiplication rule for probability calculates the probability that two events occur at the same time.
- Instruction includes connecting set operations (MA.912.LT.5.4) and exploring relationships using Venn Diagrams (MA.912.LT.5.5) when working to define the multiplication rule for probability: $P(A \text{ and } B) = P(A) \cdot P(B|A)$ or $P(A \text{ and } B) = P(B) \cdot P(A|B)$.



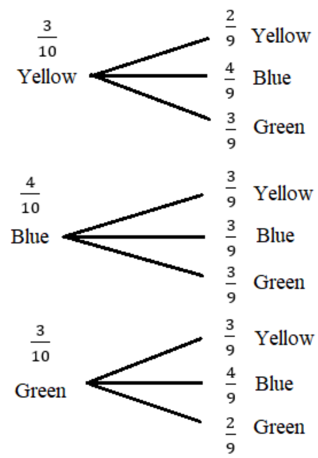
- This can also be described as $P(A \cap B) = P(A) \cdot P(B|A) = P(B) \cdot P(A|B)$.
- The probability of two events occurring at the same time leads to finding the probability one event occurs and the probability a second events occurs, given the first one already has occurred. This involves conditional probability (MA.912.DP.4.3), which leads us to the multiplication rule for probability.
    - Rearrange the conditional probability formula below to derive the multiplication rule for probability.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

      Multiply both sides of the equation by $P(B)$ to obtain $P(A \cap B) = P(B) \cdot P(A|B)$
- Instruction includes a variety of real-world examples where students must determine whether the events are independent. In doing so, develop the alternate, simplified formula for finding the probability of two events occurring.
    - If events are independent, then the probability of the first event no longer affects the probability of the second event occurring. Therefore, the formula can be simplified to $P(A \text{ and } B) = P(A) \cdot P(B)$.
- Tree diagrams can also be used to assist students with visualizing the multiplication rule for probability.
    - For example, given the box below with smiley-face erasers, find the probability of choosing a yellow eraser then a blue eraser, without replacement.

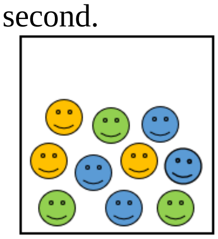


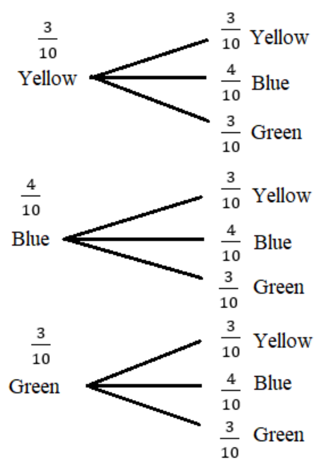
        - Use the tree diagram below to determine choosing a yellow eraser and then choosing a blue eraser.



$$P(yellow \text{ and } blue) = P(yellow) \cdot P(blue|yellow)$$
$$P(yellow \text{ and } blue) = \frac{3}{10} \cdot \frac{4}{9} = \frac{2}{15}$$

- Using a similar example, determine the probability when the events are independent. Given the box below with smiley-face erasers, find the probability of choosing 2 yellow erasers in a row, if the first eraser chosen is replaced in the box before choosing the second.





$$P(yellow\ and\ yellow) = P(yellow) \cdot P(yellow)$$
$$P(yellow \text{ and } yellow) = \frac{3}{10} \cdot \frac{3}{10} = \frac{9}{100}$$

Common Misconceptions or Errors

- Students may not be able to determine whether two events are independent.
- Students may incorrectly calculate conditional probability.

Instructional Tasks

Instructional Task 1 (MTR.4.1)

In your group take the deck of cards provided, and investigate the following outcomes:

Part A. Determine how many total cards are in the deck.

Part B. Determine the probability of drawing a heart out of your deck of cards.

Part C. Determine the probability of drawing a 4 out of your deck of cards.

Part D. Draw a 4 out of your deck, and do not put it back in to the deck. What would now be the probability of drawing a 6? Are these events independent? Explain.

Part E. Determine the probability of drawing a 4 out of your deck of cards, then without replacing it, drawing a 6.

Part F. What is the probability of drawing a face card, then without replacing it, drawing a second face card? Are these events dependent or independent? Justify your answer.

Part G. What is the probability of drawing a face card, then replacing it, drawing a second face card? Are these events dependent or independent? Justify your answer.

Instructional Items

Instructional Item 1

If flipping a fair coin twice, what is the probability of landing on heads and then tails?

Instructional Item 2

A phone has 20 downloaded songs, 8 country and 12 rock. If the downloaded songs are played on random selection, without repeating, what is the probability the first two songs chosen will be rock?

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.DP.4.9

Benchmark

MA.912.DP.4.9	Apply the addition and multiplication rules for counting to solve mathematical and real-world problems, including problems involving probability.

Connecting Benchmarks/Horizontal Alignment

Terms from the K-12 Glossary

Vertical Alignment

Previous Benchmarks

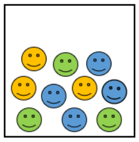
- MA.7.DP.2.2, MA.7.DP.2.3, MA.7.DP.2.4
- MA.8.DP.2.2, MA.8.DP.2.3

Next Benchmarks

Purpose and Instructional Strategies

In grade 8 and Algebra I Honors, students used organized lists, tables and tree diagrams to determine sample spaces for repeated experiments and determined sample spaces to help organize information when constructing relative frequency tables. In Math for College Liberal Arts, students consider new ways to organize data when solving real world problems involving the addition and multiplication rules for counting.

- Instruction includes determining mutually exclusive events and using the addition rule for counting with multiple events when there are no common outcomes. Given events A and B are mutually exclusive events, where there are m outcomes of event A and n outcomes of event B, then there are $m + n$ outcomes of either events A or B.
    - For example, when going to lunch at a local deli, Brendi has decided she would like either soup or a sandwich. The deli has 3 soup options and 5 sandwich options. Therefore, Brendi has 8 lunch options of soup or a sandwich.
- Instruction includes connecting the addition rule for counting with using combinations (MA.912.DP.4.10).
    - For example, from the box shown below with smiley-face erasers, determine how many ways we can choose 2 of the same color eraser.



This can be calculated using combinations for the total number of ways to choose 2 yellow erasers from the 3 yellow erasers, 2 blue erasers from the 4 blue erasers and 2 green erasers from the 3 green erasers.

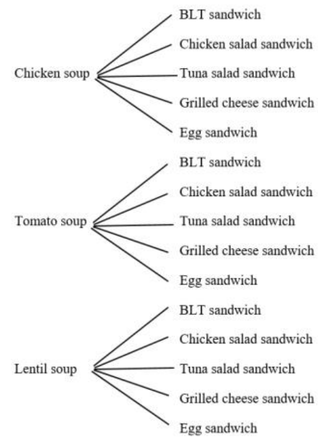
$${}_3C_2 + {}_4C_2 + {}_3C_2$$

- The multiplication rule for counting, also known as the Fundamental Counting Principle, is needed when at least two events must occur. Given independent events A and B, where there are m outcomes of event A and n outcomes of event B, then there are $m \times n$ outcomes events A and B can occur.
    - For example, when going to lunch at a local deli, Brendi has decided she would like soup and a sandwich. The deli has 3 soup options and 5 sandwich options. Therefore, Brendi has 15 different lunch options of soup and a sandwich.
- Lists or tree diagrams may be useful in illustrating this principle.
    - List

Chicken soup	BLT sandwich
Chicken soup	Chicken salad sandwich
Chicken soup	Tuna salad sandwich
Chicken soup	Grilled cheese sandwich
Chicken soup	Egg sandwich
Tomato soup	BLT sandwich
Tomato soup	Chicken salad sandwich
Tomato soup	Tuna salad sandwich
Tomato soup	Grilled cheese sandwich
Tomato soup	Egg sandwich
Lentil soup	BLT sandwich
Lentil soup	Chicken salad sandwich
Lentil soup	Tuna salad sandwich

Lentil soup	Grilled cheese sandwich
Lentil soup	Egg sandwich

- Tree diagram



- Mathematical and real-world problems for this benchmark should include 2 or more events.

Common Misconceptions or Errors

- When using a tree diagram to count, students may create a separate tree for each possibility, rather than one tree branching out to each choice.
- Students may miss or double count choices if trying to use a list versus the rule.
- Students may confuse when addition versus multiplication is appropriate.
- Students may incorrectly believe the addition and multiplication rules for counting only apply to two events.

Instructional Tasks

Instructional Task 1 (MTR.5.1)

Giselle is having a 16th birthday party that includes a dessert bar. The bakery catering the dessert bar has a variety of cakes, cupcakes, pies, cookies, and brownies.

Part A. If the bakery has choices of 7 different cakes, 15 different cupcakes, 6 different pies, 12 different cookies and 3 different brownies, how many choices of desserts does Giselle have for her dessert bar?

Part B. Giselle has decided she only wants cupcakes, cookies and brownies since they can easily be made as single servings. How many different desserts does she have to choose from now?

Part C. Mandy is going to the party but doesn't like cupcakes. How many dessert choices does she have? If Mandy decides she is going to have both a brownie and a cookie, how many possible combinations of desserts can she choose from to do so?

Part D. Joey is also going and loves all desserts. If he decides to have a cupcake, a cookie and a brownie, how many combinations of desserts does Joey have to choose from? How many combinations of desserts does Joey have to choose from if he wants two of each?

Part E. Using only pies and brownies from the bakery, find a way to have 23 combinations of different desserts.

Instructional Items

Instructional Item 1

Alexi is shopping at an electronics store for a new gaming system. Company P has 2 systems available, Company X has 3 systems available and Company N has one system available. Determine the total number of gaming systems from which Alexi can choose.

Instructional Item 2

Anderson is packing for vacation and included 5 shirts, 4 pairs of pants, 2 pairs of shoes and 3 hats. How many different outfits (shirt, pants, shoes and hat) will Anderson have on vacation?

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.DP.4.10

Benchmark

MA.912.DP.4.10 Given a mathematical or real-world situation, calculate the appropriate permutation or combination.

Connecting Benchmarks/Horizontal Alignment

Terms from the K-12 Glossary

Vertical Alignment

Previous Benchmarks

- MA.8.DP.2.1
- MA.912.DP.3.2

Next Benchmarks

Purpose and Instructional Strategies

In grade 8, students used organized lists, tables and tree diagrams to determine sample spaces for repeated experiments. In Algebra I, students considered combinations of solutions in reference to two-variable inequalities and constraints on systems of equations and inequalities. Algebra I Honors extended the grade 8 strategies for determining sample spaces to help organize information when constructing relative frequency tables. In Math for College Liberal Arts, students consider new ways to organize data when calculating permutations and combinations.

- Instruction includes explorations of real-world situations requiring counting. Begin with an example where students are able to easily write out all of the possible combinations and count, or use visual representations such as a tree diagram (*MTR.2.1*). Then move toward examples where counting becomes more cumbersome or time-consuming, generating an interest to find a more efficient method to calculate.
- Instruction introduces the concept of Fundamental Counting Principle. This includes the idea that if you can choose one item from a group of M items and a second item from a group of N items, then the total number of two-item choices is $M \times N$. In other words, the number of ways in which a series of successive things can occur is found by multiplying the number of ways in which each thing can occur.
- Depending on a student's pathway, this may be the first time students interact with

factorials. Begin with an example that will allow students to make sense of this concept. n factorial is the product of all the positive integers less than or equal to a non-negative integer, n. This is represented by $n!$.

- For example, at graduation, the valedictorian, salutatorian, and student council president will all stand on stage until it is time to hand out diplomas. How many possible seating arrangements will accommodate this honor?
- Connect students' visual representations and counts to 3!.
  - A students' visual representation could show the following:

Seat 1	**Seat 2**	**Seat 3**
VAL	SAL	SC Pres
VAL	SC Pres	SAL
SAL	VAL	SC Pres
SAL	SC Pres	VAL
SC Pres	VAL	SAL
SC Pres	SAL	VAL

- Students could also represent the situation as 3!. This means that Seat 1 has 3 choices, Seat 2 then has 2 choices, and Seat 3 has 1 choice. Students can show that 3! is equivalent to $3 \times 2 \times 1$, representing the 6 possible seating arrangements described in the table above.

- Instruction includes open-ended questions where students must decide for themselves if order matters or if repetition is acceptable (*MTR.4.1*). This allows exploration of both permutations and combinations.
  - For example, in a group of 7 students, 3 will win an award for most participation. Students can then discuss if they believe this should be given as 1^{st}, 2^{nd} or 3^{rd} place, or if 3 students are chosen and given the same award. Have students explore different options, explain their reasoning and work toward developing appropriate "rules" for the different scenarios.
- As students explore the concepts, instruction includes naming a permutation as an arrangement of a subset of a particular size (r) taken from a set of a certain size (n), with regard to the order of the arrangement, while a combination is a selection of a subset of a particular size (r) taken from a set of a certain size (n), without regard to the order/arrangement of the selection. Students should explore these concepts by showing the different permutations and combinations and see how the number of arrangements and groupings compare.
  - For example, when arranging the letters A, B and C, ABC is a different permutation than BCA. However, ABC is considered the same combination as BCA.
- Instruction includes various representations/symbols for calculating permutations and combinations. Once formulas are established, use mathematical context to ensure students can accurately use technology in these calculations.
  - ${}_nP_r = P(n, r) = \frac{n!}{(n-r)!}$ where n is the number of objects in a set and r is the number of objects in the subset being chosen.
    - For example, a coach must choose a batting order of 9 players. He has 18 players to choose from. How many batting orders are possible?

- $_nC_r = C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$ where n is the number of objects in a set and r is the number of objects in the subset being chosen.
  - For example, a coach must choose a team of 9 players. He has 18 players to choose from. How many teams are possible?

Common Misconceptions or Errors

- Students may double count arrangements when creating displays or working outside of the formulas for permutations and combinations.
- Students may confuse when a problem is asking for a permutation or a combination. To address this misconception, emphasize that order matters for a permutation but not a combination.
- Students may not use factorials correctly.
- Students may add, rather than multiply, when calculating the number of possible arrangements. To address this misconception, provide a scenario where students can easily write out all possible arrangements, count, and compare to their original calculation.

Instructional Tasks

Instructional Task 1 (MTR.3.1)

There are 18 students on the Fierce Felines swim team.

Part A. If all 18 students compete in a practice swim prior to the official start of a meet, how many outcomes for the Fierce Felines are there?

Part B. The coach needs to choose 4 students for the freestyle relay event. How many different ways can the coach choose students for this event?

Part C. Eight swimmers from four schools are competing in the 100 individual medley. How many ways can 1^{st}, 2^{nd}, and 3^{rd} place occur for this event?

Instructional Task 2 (MTR.7.1)

Part A. Using a real-world context, write a scenario requiring the use of the combination formula and calculate its answer. What criteria in your scenario shows that this is a combination?

Part B. Using a real-world context, write a scenario requiring the use of the permutation formula and calculate its answer. What criteria in your scenario shows that this is a permutation?

Part C. Exchange your scenarios with a partner and calculate the answers. Did your answers match?

Instructional Items

Instructional Item 1

There are 7 seniors this year on the soccer team, but only two can be named co-captains for the season. How many ways can the co-captains be assigned by the coach?

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.DP.4 *Develop an understanding of the fundamentals of propositional logic, arguments and methods of proof.*

MA.912.LT.4.1

Benchmark

MA.912.LT.4.1 Translate propositional statements into logical arguments using propositional variables and logical connectives.

Connecting Benchmarks/Horizontal Alignment

- MA.912.DP.4.1
- MA.912.LT.5.4, MA.912.LT.5.5, MA.912.LT.5.6

Terms from the K-12 Glossary

Vertical Alignment

Previous Benchmarks

Next Benchmarks

- MA.912.LT.4.6, MA.912.LT.4.8

Purpose and Instructional Strategies

In Math for College Liberal Arts, students translate propositional statements into logical arguments using propositional variables and logical connectives.

- This benchmark is an introduction to the fundamentals of propositional logic.
- Instruction includes the definition of propositional statements, discussion of the types of propositional statements, as well as examples of sentences that are not propositional statements.
- The building blocks of logic are propositional statements, which are statements that are either true or false. Every propositional statement in logic is either true or false. It cannot be both or neither.
  The following are examples of propositional statements.
  - The answer to 2 plus 2 is 7.
  - The capital of Florida is Tallahassee.
  - The floor is lava.
- Sentences that express opinions, questions, commands or whose truth value cannot be determined are not statements within the study of logic.
  The following are NOT propositional statements:
  - Are we there yet? (question)
  - The sky is beautiful. (opinion)
  - Take out the trash. (command)
  - This sentence is false. (truth value cannot be determined)
- Propositional statements are designated with lower-case variables. The most commonly used variables in logic are p, q, and r.
  - The structure of the statement is defined as:
    - p: The moon is made of cheese.

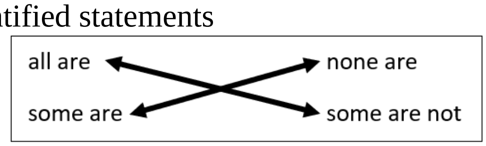
        - q: All cows have feathers.
- Propositional statements may be *simple*, *quantified* or *compound*.
    - A *simple* propositional statement is a declarative statement that does not contain a connective.
        - p: The TV is on.
    - A *quantified* propositional statement contains a quantifier such as, *all*, *none*, *or some*.
        - p: None of the remotes work.
        - q: Some of the grass is green.
    - A *compound* propositional statement includes multiple simple and quantified statements linked with logical connectives, such as 'and,' 'or,' "if…then" and "if and only if."
        - p: The car is broken and I have to go to work.
        - q: If the sun is out, then it is raining.
- Instruction includes the representation of logical connectives using logical symbolism and the use of proper terminology. Logical connectives are used to modify statements to produce new forms of statements, such as a negation, conjunction, disjunction, conditional, and biconditional.

Statement	**Key Word**	**Symbol**
negation	not	$\sim$
conjunction	and	$\wedge$
disjunction	or	$\vee$
conditional	if…then	$\rightarrow$
biconditional	if and only if	$\leftrightarrow$

    - Logical modifiers and connectives will look like the example below:
        - p: The light is on.
        - q: The door is locked.

$\sim p$	The light is not on.
$p \wedge q$	The light is on and the door is locked.
$p \vee q$	The light is on or the door is locked.
$p \rightarrow q$	If the light is on, then the door is locked.
$p \leftrightarrow q$	The light is on, if and only if the door is locked.

- Conditional statements $(p \rightarrow q)$contain a hypothesis (p) and a conclusion (q). In a conditional statement, p is the antecedent and q is the consequent.
- Negating a sentence changes its truth value; a true statement becomes false, and a false statement becomes true.
- Negating quantified and compound statements will require additional rules of logic.
    - Negating quantified statements



        - p: All birds are blue.
        - $\sim p$: Some birds are not blue.
    - Negating conjunction and disjunction statements use DeMorgan's Laws (MA.912.LT.5.6):

$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$

        - p: The car is broken.
        - q: I have to go to work.
        - $p \land q$: The car is broken and I have to go to work.
        - $\sim(p \land q)$: It is not true that, the car is broken and I have to go to work.
        - $\sim p \lor \sim q$: The car is not broken or I don't have to go to work.
    - Negating conditional statements can be thought of as breaking a promise.

$$\sim(p \rightarrow q) \equiv p \land \sim q$$

        - p: The car is broken.
        - q: I have to go to work.
        - $p \rightarrow q$: If the car is broken, then I have to go to work.
        - $\sim(p \rightarrow q)$: If it is not true that, if the car is broken, then I have to go work.
        - $p \land \sim q$: The car is broken and I don't have to go to work.
- For mastery of this benchmark, students should be given opportunities to translate a variety of propositional statements into logical symbolism and vice versa while engaging in discussions that reflect on mathematical thinking of self and others (*MTR.4.1*).
- Instruction includes connections to set operations (MA.912.LT.5.4) and Venn Diagrams (MA.912.LT.5.5).

Common Misconceptions or Errors

- Students may improperly negate quantified statements.
    - For example, when asked to negate "All giraffes are orange," students will respond with "No giraffes are orange."

  To help address this, remind students that to negate a quantified statement of the type "All A are B," we need to produce a counter-example of at least one A that is not B, or the shorter version "Some A are not B." On the other hand, to negate "No A are B," we need to find an example of an A that is also B, or "Some A are B."
- Students may improperly translate conditional statements where the hypothesis follows the conclusion statement.
    - For example, let p: "The grass is green," let q: "It is spring." Students may incorrectly translate "the grass is green if it is spring" as $p \rightarrow q$.

  To help address this, remind students that $p \rightarrow q$ translates as "if p, then q" or "p implies q." However, in the conditional statement provided, q is the hypothesis for the conclusion p. So the symbolic translation should be $q \rightarrow p$.
- Students may struggle when negating compound statements, such as conjunction, disjunction, and conditional.

Instructional Tasks

Instructional Task 1 (MTR.4.1)

For the simple statements:

p: It is a frog.
q: It has feathers.

Part A. For each compound statement, write the symbolic form.

Symbolic Form	Compound Statement
	It is a frog or it does not have feathers.
	If it is not a frog, then it does not have feathers.
	It is not a frog if it has feathers.
	It is not a frog if and only if it has feathers.

Part B. Justify your answers in writing by explaining the process you used to translate each compound statement. Compare your reasoning with others in your group.

Instructional Task 2 (MTR.1.1)

Part A. Define p as a quantified statement of your choice.
Part B. Define q as a simple statement of your choice.
Part C. Write the compound statement $(p \wedge q)$ using your definitions.
Part D. Write the compound statement $\sim p \rightarrow q$ using your definitions.

Instructional Items

Instructional Item 1

Select the negation of: *Some planets are habitable*.

a. All planets are habitable.
b. Some planets are not habitable.
c. No planets are habitable.

Instructional Item 2

For simple statements:

p: This book is boring.
q: I need coffee.

Write each of the following as a grammatically correct sentence:

- p and q
- p or q
- $\sim p \vee q$
- $\sim (p \wedge q)$
- $p \rightarrow \sim q$

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.LT.4.2

Benchmark

MA.912.LT.4.2 Determine truth values of simple and compound statements using truth tables.

Connecting Benchmarks/Horizontal Alignment

Terms from the K-12 Glossary

Vertical Alignment

Previous Benchmarks

Next Benchmarks

- MA.912.LT.4.6

Purpose and Instructional Strategies

In Math for College Liberal Arts, students determine truth values of simple and compound statements using truth tables. In other courses, students refine their knowledge and skills regarding logic and set theory.

- Building on MA.912.LT.4.1, instruction includes understanding that logic is concerned with propositional statements that are either true or false. Every propositional statement in logic has exactly one of the two truth values (either true or false, but not both or neither).
- Building on MA.912.LT.4.1, students determine when compound propositional statements involving negation, conjunction, disjunction, conditional, and biconditional, are true and when they are false. The truth-value relationships among several statements may be represented by a diagram called a *truth table*.
    - *Truth table* is a table showing the resulting truth value of a compound statement for all the possible truth values for the simple statements.

p	q	
T	T	
T	F	
F	T	
F	F	

- Instruction includes the formal definition of the basic truth tables for negation, conjunction, disjunction, conditional, and biconditional.

> Let
> p: The prize for the contest is a trip to Hawaii.
> g: The prize for the contest is a $1000.

- Negation ($\sim$) changes the truth value of a statement; a true statement becomes false, and a false statement becomes true. This can be expressed in a truth table, where T represents true and F represents false.

p	$\sim p$
T	F
F	T

  - For example, if the statement p:The prize for the contest is a trip to Hawaii is true, then the negation of this statement $\sim p$: The prize for the contest is NOT a trip to Hawaii is false.
- Conjunction ($\wedge$) is true only when both p and q are true.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- For example, the statement $\boldsymbol{p \wedge q}$: The prize for the contest is a trip to Hawaii AND \$1000 is true only when both statement p and statement q are true; otherwise, the conjunction is false.

- Disjunction (∨) is false only when both p and q are false.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

  - For example, the statement $p \vee q$: The prize for the contest is a trip to Hawaii OR \$1000 is false only when both statement p and statement q are false; otherwise, the disjunction is true.

Now let
p: I finish my homework by 9.
g: I will meet you for pizza.

- Conditional (→) is false only when the hypothesis (p) is true, but the conclusion (q) is false; this relationship may be thought of as 'breaking a promise.'

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

  - For example, $p \rightarrow q$: IF I finish my homework by 9, THEN I will meet you for pizza is false only when I do finish my homework by 9, but I do not meet you for pizza.

- Biconditional (↔) is true only when p and q have the same truth value. A biconditional is a conjunction of two conditional statements, $(p \rightarrow q) \wedge (q \rightarrow p)$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

  - For example, $\boldsymbol{p \leftrightarrow q}$: I finish my homework by 9, IF AND ONLY IF I will meet you for pizza. Biconditional can be considered as two conditional statements, $\boldsymbol{p \rightarrow q}$: If I finish my homework by 9, then I'll meet you for pizza; and $\boldsymbol{q \rightarrow p}$: If I will meet you for pizza, then I finish my homework by 9. Since the conditional is false only when the hypothesis is true, and the conclusion is false, biconditional is true only when both p and q are true, or when both are false.

- Instruction includes determining the truth value of a given compound statement using a truth table. Each operation is a different column in the truth table (see highlights in the table example below). The order of the operations starts with the inside of the parentheses.

- For example, determine the truth value of the following compound statement: $(\sim p \wedge q) \rightarrow p$. Following the order of operations, first find the negation of p, then the conjunction, and finally the conditional.
  - To complete column 3, negate column 1.
  - To complete column 4, find the conjunction of column 3 AND column 2.
  - Column 5 is an optional copy of column 1 to assist with column 6.
  - To complete column 6, find the conditional of column 4 and column 5.

Column 1	**Column 2**	**Column 3**	**Column 4**	**Column 5**	**Column 6**
p	q	$\sim p$	$(\sim p \wedge q)$	p	$(\sim p \wedge q) \rightarrow p$
T	T	F	F	T	T
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	F	F	T

Common Misconceptions or Errors

- Students may assume that in a conditional truth table, when the hypothesis is false, then the conditional is false. To address this, instruction should specify that the conditional is statement is true by default in the case when hypothesis is false, regardless of the truth value of the conclusion.

Instructional Tasks

Instructional Task 1 (MTR.4.1)

Given the statement: If today is Monday, then the unicorns are pink with purple stripes.

Part A. Is this statement true today?

Part B. When is this statement false?

Part C. Create a truth table to confirm your hypotheses in Part A and Part B.

Instructional Items

Instructional Item 1

Complete a truth table for $(p \vee \sim q) \leftrightarrow q$.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.LT.4.3

Benchmark

MA.912.LT.4.3	Identify and accurately interpret "if…then," "if and only if," "all" and "not" statements. Find the converse, inverse and contrapositive of a statement.

Benchmark Clarifications:

Clarification 1: Instruction focuses on recognizing the relationships between an "if…then" statement and the converse, inverse and contrapositive of that statement.

Clarification 2: Within the Geometry course, instruction focuses on the connection to proofs within the course.

Connecting Benchmarks/Horizontal Alignment

- MA.912.LT.5.1, MA.912.LT.5.4, MA.912.LT.5.5

Terms from the K-12 Glossary

Vertical Alignment

Previous Benchmarks

- MA.912.LT.4.3

Next Benchmarks

- MA.912.LT.5.2, MA.912.LT.5.3

Purpose and Instructional Strategies

In Geometry, students learned how to write precise and accurate definitions using "if and only if" form. In Math for College Liberal Arts, students use and think more formally about different kinds of logical statements. In other courses, students refine their knowledge and skills regarding logic and set theory.

- Instruction includes reinforcing the definition of a conditional statement in connection with MA.912.LT.4.1. In particular, a conditional statement is a statement that can be put in the form of "*if* p*, then* q," where p and q are themselves statements. The statement p is called the *hypothesis* (or the *antecedent*) of the conditional and q is called the *conclusion* (or the *consequent*). The conditional "*if* p*, then* q" is symbolized by "$p \rightarrow q$," which can be read "p *implies* q."
- In connection with the benchmark MA.912.LT.4.5, instruction includes the student understanding that a conditional statement form "*if* p*, then* q" is logically equivalent to "q *if* p." The key word "*if*" always labels the hypothesis of the conditional.
    - For example, "*I will get an A for the course if I get a perfect score on the final*" is a conditional statement that is equivalent to "*If I get a perfect score on the final, then I will get an A for the course,*" where the statement "*I get a perfect score on the final*" is the hypothesis of both forms of the conditional.
- Building on MA.912.LT.4.1 and MA.912.LT.4.5, instruction also includes discussing the connection between conditional statements and quantified statements that use "all" and "no," as well as negations (*MTR.2.1*, *MTR.4.1*).
    - For example, the quantified statement "*All surgeons are doctors*" is logically equivalent to "*If a person is a surgeon, then that person is a doctor*." The quantified statement "*No sharks are mammals*" is logically equivalent to "*If it is a shark, then it is NOT a mammal*."
- Instruction focuses on recognizing the relationships between an "if…then" statement and the converse, inverse and contrapositive of that statement.
    - For any conditional statement, there are three related statements, where the original conditional statement is transformed by rearranging or negating the hypothesis and conclusion of the conditional to create the converse, the inverse, and the contrapositive. Given a conditional statement,
        - The converse is generated by switching the hypothesis and the conclusion of the original conditional.
        - The inverse is generated by negating both the hypothesis and the conclusion of the original conditional.
        - The contrapositive is generated by negating and switching the hypothesis and the conclusion of the original conditional.

Related Statements	English Statement	Symbolic Statement
The original **conditional** is	"*if p then q*"	$p \rightarrow q$
The **converse** is	"*if q then p*"	$q \rightarrow p$
The **inverse** is	"*if not p then not q*"	$\sim p \rightarrow \sim q$
The **contrapositive** is	"*if not q then not p*"	$\sim q \rightarrow \sim p$

- A conditional statement and related statements will look like the example below.
  - *p*: There is a hurricane.
  - *q*: All restaurants are closed.

Variations of the Conditional	Symbolic Statement	English Translation
Conditional	$p \rightarrow q$	If there is a hurricane, then all restaurants are closed.
Converse	$q \rightarrow p$	If all restaurants are closed, then there is a hurricane.
Inverse	$\sim p \rightarrow \sim q$	If there is no hurricane, then some restaurants are not closed.
Contrapositive	$\sim q \rightarrow \sim p$	If some restaurants are not closed, then there is no hurricane.

- In connection to MA.912.LT.4.5, instruction includes the student understanding of the following logical equivalences:
  - A conditional statement and its contrapositive are logically equivalent.
  - The converse and inverse of a conditional statement are logically equivalent.
- Instruction focuses on recognizing the relationships between an "if…then" statement and its converse to form a "*if and only if*" biconditional statement.
  - Instruction includes reinforcing the definition of biconditional statements in connection with MA.912.LT.4.1. In particular, a biconditional statement is a statement that can be put in the form of "*p if and only if q*," where *p* and *q* are themselves statements. The biconditional "*p if and only if q*" is symbolized by "$p \leftrightarrow q$." The phrase *if and only if* can be abbreviated as *iff*.
  - A biconditional "$p \leftrightarrow q$" is a statement that be put in the form "$p \rightarrow q$" AND "$q \rightarrow p$." In other words, a biconditional is a statement that results when a conditional statement and its converse are combined in an abbreviated way.
    - For example, "A triangle is equilateral if and only if the three sides have the same length." is a compact way of saying "If the triangle is equilateral, then the three sides of the triangle have the same length." AND "If the three sides of the triangle have the same length, then the triangle is equilateral."
- Instruction includes the use of truth tables to help students organize their work and show equivalency (MA.912.LT.4.2).

Common Misconceptions or Errors

- Students may confuse the hypothesis and conclusion.
- Students may try to change the inverse, converse or contrapositive to make sense in the real world rather than using the logic definitions.

Instructional Tasks

Instructional Task 1 (MTR.2.1)

Use the statements below to identify the conditional, converse, inverse and contrapositive of the statement.

"All cats are animals."

a. If it is a cat, then it is an animal.
b. If it is not a cat, then it is not an animal.
c. If it is not an animal, then it is not a cat.
d. If it is an animal, then it is a cat.
e. It is an animal, if it is a cat.

Instructional Items

Instructional Item 1

Use the following biconditional statement to answer the questions.

A triangle is an equiangular triangle if and only if the triangle has three congruent angles.

Part A. Write the two "if then" statements that can be written from the given biconditional statement.

Part B. Write the converse of one of the conditional statements created in Part A.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.LT.4.4

Benchmark

MA.912.LT.4.4	Represent logic operations, such as AND, OR, NOT, NOR, and XOR, using logical symbolism to solve problems.

Connecting Benchmarks/Horizontal Alignment

- MA.912.DP.4.1
- MA.912.LT.5.4, MA.912.LT.5.5

Terms from the K-12 Glossary

Vertical Alignment

Previous Benchmarks

- MA.6.AR.2.1
- MA.8.F.1

Next Benchmarks

Purpose and Instructional Strategies

In Math for College Liberal Arts, students represent logical operations, such as AND, OR, NOT, NOR, and XOR, using logical symbolism to solve problems involving classifying items as belonging to sets.

- This benchmark builds on MA.912.LT.4.1 and MA.912.LT.4.2 as it involves representing or translating statements using logical connectives or operations, as well as determining the truth value of an output given the truth value of the inputs.
- Instruction includes defining of the logic operations as functions.

Logic Operation	Result	Symbol

AND	output is only true when all of its inputs are true, otherwise the output is false	$\wedge$
OR	output is only true if one or more of its inputs are true, otherwise the output is false	$\vee$
NOT	output is true when its single input is false, and false when its single input is true	$\sim$
NOR	output is only true when all of its inputs are false, otherwise the output is always false	$\downarrow$
XOR *exclusive disjunction*	output is true if one, and only one, of the inputs is true. If both inputs are false or both are true, a false output results	$\oplus$

- Operators may be in capital or lower case letters. Instruction progresses from using all capital letters to lower case letters in order to ensure students recognize the operators.
- Instruction includes connections to set operations and Venn Diagrams (MA.912.LT.5.4 and MA.912.LT.5.5) using real-world contexts such as choosing the appropriate logic operation and writing a search query to identify a subset of a population.
    - For example, suppose R is the set of all romance books, and T is the set of all true crime books in a library database.
      A search query for romance
        - All the books that are an element of the set R.

      A search query for romance AND true crime
        - All the books that are both a romance and a true crime, which is the intersection of R and T.

      A search query for romance OR true crime
        - All the books that are a romance, a true crime, or both, which is the union of R and T. This is the inclusive or.

      A search query for NOT true crime
        - All the books that are not a true crime, which is the complement of T.

      A search query for romance NOR true crime
        - All the books that are not a true crime and not a romance, which is the complement of the union of R and T.

      A search query for romance XOR true crime
        - All the books that are a true crime or a romance, but not both. This is the union minus the intersection, which is the exclusive or.
- Instruction directs students to decide whether an element is a member of resulting set given the conditions and logic operations (*MTR.6.1*).
    - For example, if given a book that is a true crime romance, students would determine to which of the operator(s) AND, OR, NOT, NOR or XOR it belongs.
    - For example, if given an operator of NOT, does the true crime romance book belong to that set.
- Students also engage in discussions with their peers, analyzing the mathematical thinking of others (*MTR.4.1*), demonstrating their understanding by representing their reasoning in multiple ways (e.g., Venn Diagram, set operations, specific examples of set elements).
- Instruction mentions that logic operators are used in Boolean Algebra, logic gates and

computer science topics.
  - Boolean Algebra focuses on binary variables with truth values being coded as a 0 or 1.
  - Logic Gates is the visualization model of Boolean Algebra. Logic gates can be a device that acts as building blocks for digital circuits.

Common Misconceptions or Errors

- Students may use the logic operation OR as 'exclusive or.' To help address this, make connection to the set operation of the union to reinforce the meaning of 'p OR q' as 'p or q or both p AND q.'
- Students may use the logic operation AND to join two sets. To help address this, make connection to the set operation of the intersection.

Instructional Tasks

Instructional Task 1 (MTR.1.1, MTR.4.1)

Work with a partner(s) to determine the set of numbers that meet these conditions as a whole.

Part A. What are some examples of numbers that meet the condition "NOT an odd number?"

Part B. What are some examples of numbers that meet the condition "less than 20 AND greater than 1?"

Part C. What are some examples of numbers that meet the condition "multiple of 3 OR multiple of 5?"

Part D. Describe the numbers that meet the following conditions: NOT an even number AND less than 20 AND greater than 1 AND (multiple of 3 OR multiple of 5).

Instructional Items

Instructional Item 1

Suppose your high school has a database of all students along with many characteristics of each student such as name, current grade level, intended career path, and many others. A segment of the student characteristics table is shown.

ID	Last_Name	First_Name	Grade_Level	Career_Path
113344	Ghuft	Ben	9	Undecided
223344	Mezner	Lizzy	9	STEM
445566	Lorg	Ian	10	STEM
778899	Resnick	Tim	12	Business

Which of the following search queries would generate the list of 12^{th} graders, on a Business Career Path?

a. Grade_Level is 12 AND Career_Path is Business

b. Grade_Level is greater than 11 OR Career_Path is Business

c. Grade_Level is NOT 9 AND Grade_Level is NOT 10 AND Grade_Level is NOT 11 AND Career_Path is Business

d. Grade_Level is 12 AND Career_Path is NOT STEM

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.LT.4.5

Benchmark

MA.912.LT.4.5 Determine whether two propositions are logically equivalent.

Connecting Benchmarks/Horizontal Alignment

- MA.912.LT.5.1, MA.912.LT.5.6

Terms from the K-12 Glossary

- Equation

Vertical Alignment

Previous Benchmarks

- MA.912.AR.1

Next Benchmarks

Purpose and Instructional Strategies

In Math for College Liberal Arts, students determine whether two propositions are logically equivalent using the truth values of the statements. In other courses, students will continue reinforcing the concept of equivalence and equivalent relationships.

- Students have encountered the concept of 'equivalence' throughout their K-12 education.
  - For instance, in grade 6, students applied properties of operations to rewrite numbers in equivalent forms. In grade 7, students rewrote algebraic expressions in equivalent forms.
- Instruction includes the discussion of intuitive understanding of equivalence given concrete examples.
  - For example, students are asked to discuss whether the propositional statement "*It is false that the wall is not green*" is logically equivalent to the statement "*The wall is green*" (*MTR.4.1*).
  - In connection with MA.912.LT.4.1, students may be guided to translate the above example into symbolic statements to argue for equivalence:

    $\sim(\sim p)$: It is false that the wall is not green.

    p: The wall is green.
- Instruction includes the formal definition of equivalent statements: two propositions are logically equivalent if they always have the same truth values.
  - In particular, equivalent compound statements are composed of the same simple statements and have the same corresponding truth values for any given combination of the truth values of these simple statements. In other words, if a compound statement is true, then its equivalent statement is true; if a compound statement is false, then its equivalent statement is false.
- A special symbol, $\equiv$, is used to show that two statements are equivalent.
  - For example, $p \equiv \sim(\sim p)$ signifies that a statement p is logically equivalent to the negation of the negated p. The symbol $\equiv$ means "is equivalent to."
- If two statements are not equivalent, this can be shown with $\not\equiv$.
  - For example, $p \not\equiv \sim p$ signifies that a statement p is not logically equivalent to its negation.
- In connection with MA.912.LT.4.2, instruction guides students to construct the truth tables to show that two propositional statements are equivalent.
  - For example, ask students to demonstrate that $p \equiv \sim(\sim p)$ by constructing a truth

table showing that the corresponding truth values are the same:

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

Equivalent (same corresponding truth values)

- In connection with MA.912.LT.4.5, instruction reinforces the student understanding of the following logical equivalences using truth tables:
  - A conditional statement and its contrapositive are logically equivalent.
    - $p \rightarrow q \equiv \sim q \rightarrow \sim p$
  - The converse and inverse of a conditional statement are logically equivalent.
    - $q \rightarrow p \equiv \sim p \rightarrow \sim q$

p	q	Conditional $p \rightarrow q$	Converse $q \rightarrow p$	Inverse $\sim p \rightarrow \sim q$	Contrapositive $\sim q \rightarrow \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Equivalent

- Connecting MA.912.LT.4.1 and MA.912.LT.5.6 to instruction includes the verification and reinforcement of the equivalent relationships discussed in prior or upcoming work.
  - DeMorgan's Laws
    The compound statement "$\sim(p\ AND\ q)$" is logically equivalent to "$(\sim p)\ OR\ (\sim q)$."
    - $\sim(p \wedge q) \equiv \sim p \vee \sim q$

    The compound statement "$\sim(p\ OR\ q)$" is logically equivalent to "$(\sim p)\ AND\ (\sim q)$."
    - $\sim(p \vee q) \equiv \sim p \wedge \sim q$

    Instruction guides students to verify the DeMorgan's Laws using a truth table. For instance, students are guided to see from the truth table for $\sim(p \wedge q)$ & $\sim p \vee \sim q$ that these statements are equivalent.

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Equivalent

Common Misconceptions or Errors

- When the wording of a problem prompt is written as "show that statement p and statement q are equivalent," students may incorrectly interpret "and" as a conjunction when constructing a truth table.

Instructional Tasks

Instructional Task 1 (*MTR.1.1*, *MTR.2.1*, *MTR.3.1*)

Given two statements: $p \rightarrow q$ and $\sim p \vee q$.

Part A. Show that $p \rightarrow q$ is equivalent to $\sim p \vee q$.

Part B. Use the result from part A to write a statement that is equivalent to: "Do not wash that sweater or it will be ruined."

Part C. Given that $p \rightarrow q \equiv \sim p \vee q$, apply DeMorgan's Law to demonstrate that $\sim (p \rightarrow q) \equiv p \wedge \sim q$. Support your work by constructing a truth table.

Instructional Items

Instructional Item 1

Select the statements that are equivalent to

"If it is a car, then it is blue."

a. If it is not a car, then it is not blue.
b. If it is not blue, then it is not a car.
c. It is not a car or it is blue.
d. It is a car or it is not blue.
e. It is not a car and it is blue.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.LT.4.9

Benchmark

MA.912.LT.4.9 Construct logical arguments using laws of detachment, syllogism, tautology, contradiction and Euler Diagrams.

Connecting Benchmarks/Horizontal Alignment

- MA.912.LT.5.5

Terms from the K-12 Glossary

Vertical Alignment

Previous Benchmarks

- MA.912.GR.1
- MA.912.GR.5

Next Benchmarks

Purpose and Instructional Strategies

In Geometry, students learned to construct proofs for many of the geometric facts that they encountered and learned to use counterexamples to check the validity of arguments and statements. In Math for College Liberal Arts, students learn to construct logical arguments using valid and invalid argument forms, truth tables to form tautologies and Euler Diagrams.

- Instruction includes the definition of a logical argument. A logical argument consists of given statements, called *premises*, and a *conclusion*. The premises present evidence or reasons in support of the conclusion.
    - In English, conclusion-indicator words include *therefore*, *thus*, *hence*, *consequently*, *so* and others that indicate that a conclusion is being made. Once the conclusion statement is identified, the remaining statements are considered premises.

- In symbolic form, the three-dot triangle, $\therefore$, indicates the conclusion. Moreover, a bar, ———, is used to separate premises from the conclusion, much like an equal sign in a vertical addition problem.
  - Making a connection to MA.912.LT.4.1, instruction presents an example of an argument composed of two premises and a conclusion, where simple statements p, q, & r are defined as follows:

    p: I upgrade my computer.

    q: My computer will run faster.

    r: I will be more productive.

Argument A	**English**	**Symbolic Form**
Premise 1	*If I upgrade my computer, then it will run faster.*	$p \to q$
Premise 2	*If my computer runs faster, then I will be more productive.*	$q \to r$
Conclusion	*Therefore, if I upgrade my computer, then I will be more productive.*	——— $\therefore\ p \to r$

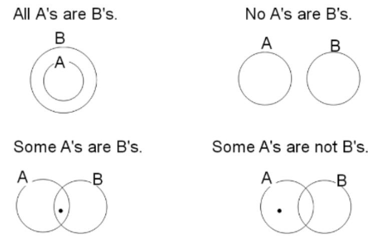
- Law of Detachment, also known as direct reasoning, allows you to "detach" the premise from the conclusion. The law provides the truth value of conditional statements. If the conditional statement and the premise are true, then the conclusion is also true. The Law of Detachment states that if $p \to q$ is true and p is true, then q is true.
  - For example, if $p \to q$ is: *If I upgrade my computer, then it will run faster* and this statement is true, as well as the statement "I upgrade my computer" is true, then using the Law of Detachment, I can conclude, "My computer will run faster" is true.
- A tautology is a compound statement that is always true, regardless of the truth values of its individual statements.
  - For example, the compound statement *It will rain today, or it will not rain today* is always true.
- Law of Syllogism states that if $p \to q$ and $q \to r$, then $p \to r$ is true. This logic is similar to transitive property of equality referenced in Appendix D.
  - An example for Law of Syllogism is shown below.

    If $p \to q$ is: *If I upgrade my computer, then it will run faster*.

    And $q \to r$ is: *If my computer runs faster, then I will be more productive.*

    Then $p \to r$ is: *If I upgrade my computer, then I will be more productive.*

    If the two statements are true, then the conclusion is also true.
- Contradiction is a compound statement that is always false.
  - For example, the compound statement *It will rain today, and it will not rain today* is always false.
- Instruction includes the definition of Euler (pronounced as "oiler") Diagrams in relation to Venn Diagrams for four types of quantified statements (*MTR.2.1*). Building on their knowledge of Venn Diagrams and quantified statements (MA.912.LT.4.1), students engage in discussion to reason about and connect multiple representations of the quantified statements.



- For enrichment of this benchmark, instruction reinforces the connection between a tautology and a contradiction: a statement that is always true, regardless of the truth values of the composite simple statements, is a tautology; whereas, a statement that is always false – is called a contradiction. Students engage in discussions to highlight the difference between the two concepts (*MTR.4.1*).

Common Misconceptions or Errors

- Students may confuse the different types of logical arguments, specifically, the Law of Detachment and Law of Syllogism.
- Students may assume that every conclusion is a valid conclusion because it is part of a given argument. To address this, help students understand that a conclusion is logical or valid only if it forms a valid argument; thus, to determine whether a conclusion is valid, we must apply the methods learned to judge the validity of a given argument in tandem with MA.912.LT.4.10.

Instructional Tasks

Instructional Task 1 (*MTR.4.1*)

Part A. Construct four logical arguments.

Part B. Create two examples of an argument using the Law of Syllogism.

Instructional Task 2 (MTR.4.1)

Is the statement "If war is peace, then war is not peace" a tautology? Is it a contradiction? Justify your answer.

Instructional Task 3 (*MTR.5.1*)

Draw two Euler diagrams to represent the possible relationships between insects and birds.

All birds have wings.
Some insects have wings.
Therefore ___________.

Instructional Items

Instructional Item 1

Construct a valid argument using the Law of Detachment given the statements.

I will go to Publix Monday.
Today is Monday.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.LT.4.10

Benchmark

MA.912.LT.4.10 Judge the validity of arguments and give counterexamples to disprove statements.

Benchmark Clarifications:
Clarification 1: Within the Geometry course, instruction focuses on the connection to proofs within the course.

Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.1.6
- MA.912.LT.5.5

Terms from the K-12 Glossary

Vertical Alignment

Previous Benchmarks

- MA.912.GR.1
- MA.912.GR.5

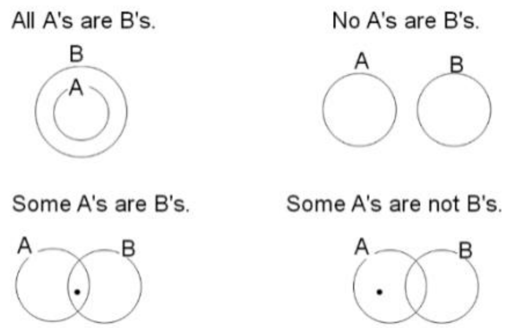
Next Benchmarks

Purpose and Instructional Strategies

In Geometry, students learned to construct proofs for many of the geometric facts that they encountered, and learned to use counterexamples to check the validity of arguments and statements. In Math for College Liberal Arts, students learn to judge the validity of logical arguments using standard argument forms, truth tables to determine tautologies and Euler Diagrams.

- This benchmark builds on MA.912.LT.4.9 from writing logical arguments to judging the validity of those arguments.
- Instruction includes the understanding that a valid argument can be a statement or sequence of statements supported by valid reasons, a part of a proof or an entire proof.
- Students should have experience determining the validity of arguments using multiple approaches, including recognizing the standard forms of valid and invalid arguments, truth tables to identify tautologies or counterexamples and Euler Diagrams; making the connection to MA.912.LT.4.9 (*MTR.5.1*).
- One way to show that an argument is not valid is to provide at least one counterexample to at least one statement in the argument. In other words, an example that proves a statement false is called a counterexample.
    - For example, if the argument is “All rectangles have opposite sides parallel; therefore, given a quadrilateral is not a rectangle, the quadrilateral does not have opposite sides parallel,” then a student can provide a parallelogram as a counterexample to show that concluding statement of the argument is not valid.
- Instruction includes the definition of a valid and invalid argument.
    - In a valid argument, conclusion must follow from the given set of premises. In other words, if the premises are true, the conclusion must also be true.
    - An argument is invalid (or a fallacy) if the conclusion does not necessarily follow from given premises. In other words, if an argument is invalid, there is a case when a conclusion is false even when all the premises are true.

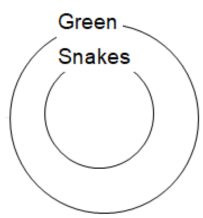
- In connection with MA.912.LT.5.5, instruction includes the definition of Euler Diagrams in relation to Venn Diagrams for four types of quantified statements (*MTR.2.1*). Building on their knowledge of Venn Diagrams and quantified statements (MA.912.LT.4.1), students engage in discussion to reason about and connect multiple representations of the quantified statements.



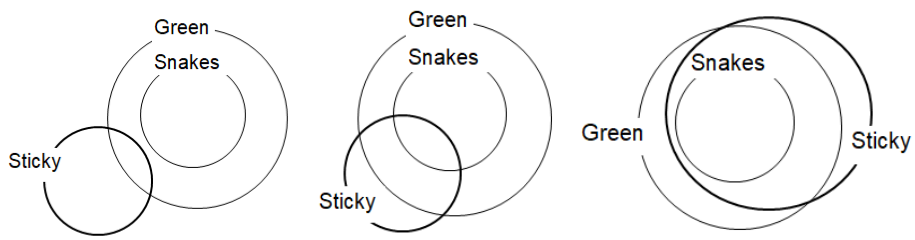
- To practice constructing Euler Diagrams, problem types will include arguments containing quantified statements.
    - Example:

      Premise 1: All snakes are green.
      Premise 2: Some green things are sticky.
      Conclusion: Therefore, some snakes are sticky.

      Step 1: Draw Premise 1



Step 2: Allow students to create as many accurate representations as possible of Premise 1 along with Premise 2.



Ask students whether any of the diagrams created DO NOT support the conclusion. Since the first diagram concludes "No snakes are sticky," the argument is invalid because this diagram contradicts the conclusion, indicating that the conclusion 'Some snakes are sticky' is not true in every case.

- Instruction includes multiple approaches to determine the validity of arguments (MA.912.LT.4.10), including recognizing the standard forms of valid and invalid arguments, truth tables to identify tautologies or counterexamples, and Euler Diagrams.
    - Approach | Valid and Invalid Argument Forms.
      If an English argument translates (MA.912.LT.4.1) into one of the standard argument forms, we can automatically determine the validity of the argument as either valid or invalid.

VALID					
Law of Detachment (i.e., Direct Reasoning)	**Contrapositive Reasoning**	**Disjunction Reasoning**		**Law of Syllogism (i.e., Transitive Reasoning)**	
$p \rightarrow q$ p ——— $\therefore q$	$p \rightarrow q$ $\sim q$ ——— $\therefore \sim p$	$p \vee q$ $\sim p$ ——— $\therefore q$	$p \vee q$ $\sim q$ ——— $\therefore p$	$p \rightarrow q$ $q \rightarrow r$ ——— $\therefore p \rightarrow r$	$p \rightarrow q$ $q \rightarrow r$ ——— $\therefore \sim r \rightarrow \sim p$

INVALID					
Fallacy of the Converse	**Fallacy of the Inverse**	**Misuse of Disjunctive Reasoning**		**Misuse of Law of Syllogism**	
$p \rightarrow q$ q ——— $\therefore p$	$p \rightarrow q$ $\sim p$ ——— $\therefore \sim q$	$p \vee q$ p ——— $\therefore \sim q$	$p \vee q$ q ——— $\therefore \sim p$	$p \rightarrow q$ $q \rightarrow r$ ——— $\therefore r \rightarrow p$	$p \rightarrow q$ $q \rightarrow r$ ——— $\therefore \sim p \rightarrow \sim r$

- For example, Argument A is valid because, when translated (see above), it matches the Law of Syllogism standard argument form:

  Argument A

  If I upgrade my computer, then it will run faster.
  If my computer runs faster, then I will be more productive.
  Therefore, if I upgrade my computer, then I will be more productive.

- Approach | Rewriting Arguments

  A given argument can be rewritten as a conditional statement, where the conjunction of its premises is the antecedent, and the conclusion is the consequent (MA.912.LT.4.3). In other words, we can rewrite an argument with two premises in the following form: *If (premise 1 and premise 2), then conclusion*.
  - This argument is valid if, (conjunction of premises) → (conclusion) is a *tautology* (a statement that is always true, regardless of the truth values of its composite statements).
  - If the (conjunction of premises) → (conclusion) is not a tautology, the argument is invalid.
- Approach | Premises Implying Conclusions

  If an argument is valid, then true premises imply a true conclusion. Therefore, if true premises do not imply a true conclusion, then the argument is not valid. This idea forms the basis of the next method where one aims to identify a row in the constructed truth table where true premises lead to a false conclusion. In other words, this approach seeks to identify a counterexample (a case that proves a statement false).

  Step 1: Express the premises and the conclusion symbolically, using a letter to represent each simple statement in the argument.

  Step 2: Make a truth table with 3 columns – one for each premise and one for the conclusion.

  Step 3: If in the truth table we see a "bad row" (all true premises followed by a false conclusion: T T F row), then the argument is invalid. If we do not see a "bad row", the argument is valid (*MTR.5.1*).
  - For example, Argument A is valid because when all premises are true,

the conclusion is also true. Therefore, there is no ‘bad row’ (TTF pattern) in the truth table:

p	q	r	$p \to q$	$q \to r$	$p \to r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

- Approach | Euler Diagrams
  For arguments, whose premises are quantified statements (i.e., statements containing the words *all*, *some*, and *no*), Euler Diagrams can be used to determine validity. In order to determine the validity of arguments using Euler Diagrams, the goal is to show the argument is invalid by producing a counterexample, or a diagram of the premises that does NOT illustrate the conclusion. If there is no such counterexample, then the argument is valid. In other words, the argument is valid if and only if every possible Euler Diagram of the premises illustrates the conclusion of the argument.
  - The Euler Diagram approach is similar to the logic of the ‘bad row’ in that it assumes that if an argument is valid, then true premises always imply a true conclusion. Therefore, if true premises do not imply a true conclusion, then the argument is not valid.

  Step 1: Make a diagram of the first premise.
  Step 2: If possible, make a diagram for the second premise so that conclusion DOES NOT follow.
  Step 3: If able to do so, then the argument is INVALID. If not, then the argument is VALID (*MTR.5.1*).

- To determine whether the conditional statement is a tautology, students are guided to construct a truth table (MA.912.LT.4.2).
  - For example, Argument A is valid because the final column in the constructed truth table for the conditional $[(p \to q) \wedge (q \to r)] \to (p \to r)$ is true in every case, which means the conditional statement is a tautology.

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

- Problem types include both valid and invalid arguments, requiring students to determine the validity using multiple approaches, as applicable. In particular, given an argument in English, students will be able to examine whether it matches any of the standard argument forms, as well as use the truth table approaches to determine the validity of the argument.

- For example, the invalid argument below, students will recognize that it matches the standard form of Misuse of Disjunctive Reasoning, as well as the truth table in conjunction with the 'bad row' to identify the case when all true premises lead to a false conclusion (T T F row), showing that the argument is invalid.

p: The bug is dead.
q: The bird is white.

Premise 1: The bug is dead or the bird is white. *Premise 2*: The bug is dead. *Conclusion*: Therefore, the bird is not white.	$p \vee q$ p $\therefore \sim q$

		Premise 1	*Premise 2*	*Conclusion*
p	q	$p \vee q$	p	$\sim q$
T	T	T	T	F
T	F	T	T	T
F	T	T	F	F
F	F	F	F	T

Common Misconceptions or Errors

- Students may think a statement is true because they cannot think of a counterexample.
- When determining the validity of syllogisms using Euler Diagram, students may incorrectly attempt to illustrate the conclusion to show that the argument is valid, as opposed to seeking to find a counterexample.
- Students may assume that every conclusion is a valid conclusion because it is part of a given argument. To address this, help students understand that a conclusion is logical or valid only if it forms a valid argument; thus, to determine whether a conclusion is valid, we must apply the methods learned to judge the validity of a given argument.
- When looking for a 'bad row,' students may incorrectly assume that a 'T T T' (i.e., true premises lead to a true conclusion) row in the constructed truth table implies that an argument is valid. To address this, help students understand that an argument is valid *only* when all true premises imply a true conclusion in *every* case. Therefore, if there is a case when true premises do not imply a true conclusion, then the argument is not valid. The goal of the process is to look for such 'T T F' case.

Instructional Tasks

Instructional Task 1(MTR.5.1)

Determine whether each argument is valid or invalid.

Argument A

If my dog is green, then my cat is purple.
If my cat is purple, then my ferret is yellow.
Therefore, if my ferret is not yellow, then my dog is not green.

Argument B

Some pigs are brown.
Some dogs are brown.
Therefore, all dogs are pigs.

Argument C

$$\sim p \vee q$$
$$p \leftrightarrow r$$
$$\therefore p \wedge r$$

Instructional Task 2 (*MTR.5.1*)

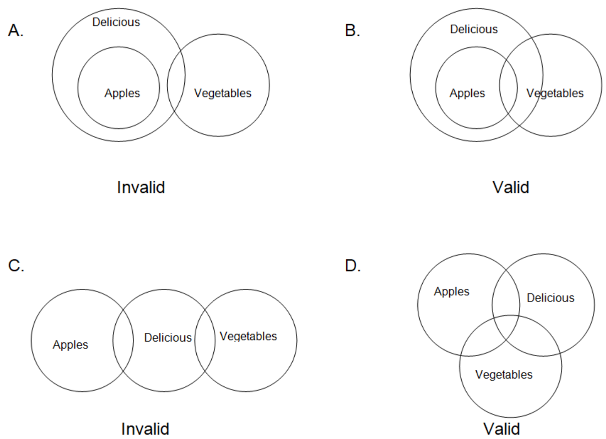
Use the standard forms of valid arguments to draw a valid conclusion from the given premises

I will go to Publix Monday or Tuesday.
I did not go to Publix Monday.
Therefore, ____________________

Instructional Task 3 (*MTR.5.1*)

Choose the correct Euler diagram used to determine the validity of this argument and state whether the argument is valid or invalid.

Some apples are delicious.
Some vegetables are delicious.
Therefore, some apples are vegetables.



Instructional Items

Instructional Item 1

Fill in each blank so that the resulting statement is true.

The argument

$$p \vee q$$
$$q$$
$$\therefore \sim p$$

is called Misuse of _________ and is __________ because $[(p \vee q) \wedge (q)] \rightarrow (\sim p)$ is not a tautology.

MA.912.LT.5.1

Benchmark

MA.912.LT.5.1 Given two sets, determine whether the two sets are equivalent and whether one set is a subset of another. Given one set, determine its power set.

Connecting Benchmarks/Horizontal Alignment

Terms from the K-12 Glossary

Vertical Alignment

Previous Benchmarks

Next Benchmarks

- MA.912.LT.5.3

Purpose and Instructional Strategies

In Math for College Liberal Arts, students begin to learn about sets and subsets and their equivalency. In other classes, students will explore additional information about equivalency of sets.

- Instruction includes an introduction to sets. A set is a collection of objects. The members of a set are called elements. Sets are represented by capital letters.
- Sets can be described in three ways.
  - Word Description: W is the set of days of the week.
  - Roster Form: elements are listed in { }. The order of the elements does not matter. $W = \{Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday\}$
  - Set Builder Notation: $\{x|x\ is\ \_\_\_\_\_\}$.

    This is read "the set of all x such that x is ________."

    $W = \{x|x\ is\ a\ day\ of\ the\ week\}$

  The empty set, or null set, is a set with no elements and can be represented by { } or $\emptyset$.
- Instruction includes defining equivalent sets as sets with the same number of elements (same cardinality $n(A) = n(B)$).
  - Example:

    Given set $A = \{a, b, c, d\}$ and set $B = \{1,2,3,4\}$ sets A and B are equivalent because both sets have four elements.
- Instruction includes defining a subset as a set whose elements are all elements of another set: $A \subseteq B$ if all elements of A are also in B or there is no element in A that is not in B.
  - Example:

    Given set $A = \{a, b, c\}$ and set $B = \{a, b, c, d\}$

    $A \subseteq B$ but $B \nsubseteq A$

    The empty set is a subset of every set – there is no element in the empty set that is not in the other set.
- Students will find the power set of set A, $P(A)$ which is defined as the set of all subsets of set A.
  - Example:

    Given $A = \{a, b\}$

    The power set is $P(A) = \{\{\ \}, \{a\}, \{b\}, \{a, b\}\}$

- Example:

Given $C = \{red, white, blue\}$

The power set is $P(C) =$

$\{\{\,\}, \{red\}, \{white\}, \{blue\}, \{red, white\}, \{red, blue\}, \{white, blue\}, \{red, white, blue\}\}$

Common Misconceptions or Errors

- Students may not include the empty set or the set itself in the power set.

Instructional Tasks

Instructional Task 1 (MTR.4.1)

Given the following sets:

$$A = \{2,4,6,8,10\}$$
$$B = \{1,3,5,7,9\}$$
$$C = \{6,4,8,2\}$$
$$D = \{a, b, c, d, e\}$$
$$E = \{a, b, c\}$$
$$F = \{10,8,6,4,2\}$$

Part A. Identify the sets that are equivalent to set A.

Part B. Identify the sets that are subsets of A.

Instructional Task 2 (MTR.5.1)

Part A. Fill in the chart below.

Set	Number of Elements	Subsets	Number of Subsets
$\{\,\}$			
$\{\triangle\}$			
$\{\triangle, \diamond\}$			
$\{\triangle, \diamond, \square\}$			

Part B. How many subsets would you expect there to be for the set $\{red, white, blue, green\}$?

Part C. How many subsets would you expect there to be for the set $\{Josh, Abe, Jonah, Allie, Adam, Shane\}$?

Part D. Write an equation to represent how many subsets there are for a set of n elements.

Instructional Items

Instructional Item 1

Find the power set of set A if $A = \{a, b, c, d\}$.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.LT.5.4

Benchmark

MA.912.LT.5.4 Perform the set operations of taking the complement of a set and the union, intersection, difference and product of two sets.

Benchmark Clarifications:
Clarification 1: Instruction includes the connection to probability and the words AND, OR and NOT.

Connecting Benchmarks/Horizontal Alignment

- MA.912.DP.4.1, MA.912.DP.4.4
- MA.912.LT.4.4

Terms from the K-12 Glossary

Vertical Alignment

Previous Benchmarks

Next Benchmarks

Purpose and Instructional Strategies

In Math for College Liberal Arts students will learn to perform set operations including complement, union, intersection, difference and product of two sets. In later classes, students will continue working with these operations.

- Instruction includes connection to MA.912.LT.5.5 and using Venn Diagrams to help students visualize set operations.
- Instruction includes discussion of a universal set U which contains all the possible elements of a set.
- Instruction includes finding the complement of a set denoted by the word NOT. The complement of a set A is all elements in the universal set that are not in the set A. The complement is written A'. Other benchmarks may refer to the complement set as A^c, $\bar{A}$, A' or $\sim A$.
    - Example:
      Given $U = \{1,2,3,4,5,6,7\}$ and $A = \{2,4,6\}$, $A' = \{1,3,5,7\}$.
- Instruction includes finding the union of two sets denoted by the word OR. The union of sets A and B is the set of all elements in set A OR in set B OR in both sets. The union of sets A and B is written $A \cup B$.
    - Example:
      Given $A = \{1,2,3,4,5\}$ and $B = \{2,4,6,8\}$, $A \cup B = \{1,2,3,4,5,6,8\}$.
- Instruction includes finding the intersection of two sets denoted by the word AND. The intersection of sets A and B is the set of all elements in both set A AND set B. The intersection of sets A and B is written $A \cap B$.
    - Example:
      Given $A = \{1,2,3,4,5\}$ and $B = \{2,4,6,8\}$, $A \cap B = \{2,4\}$.
- Instruction includes finding the difference of two sets. The difference between sets A and B is the set of all elements in set A that are not in set B. The difference of sets A and B is written $A - B$. Note that order is important when finding a difference.
    - Example:
      Given $A = \{1,2,3,4,5\}$ and $B = \{2,4,6,8\}$, $A - B = \{1,3,5\}$.

- Example:
  Given $A = \{1,2,3,4,5\}$ and $B = \{2,4,6,8\}$, $B - A = \{6,8\}$.

- Instruction includes finding the product of two sets. The product of sets A and B is the set of ordered pairs (a, b) where a is an element of set A and b is an element of set B. The product of sets is sometimes called the Cartesian product or cross product. The product of sets A and B is written $A \times B$. Note that order is important when finding a product.
  - Example:
    Given $A = \{a, b\}$ and $B = \{1,2,3\}$,
    $$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$
    The product can be demonstrated in table form.

$A \times B$	1	2	3
a	$(a, 1)$	$(a, 2)$	$(a, 3)$
b	$(b, 1)$	$(b, 2)$	$(b, 3)$

  - Example:
    Given $A = \{a, b\}$ and $B = \{1,2,3\}$,
    $$B \times A = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$
- Instruction makes the connection between set operations and probability.

Common Misconceptions or Errors

- Students may repeat elements that are in both sets when writing the union.
- Students may confuse union and intersection.
- Students may incorrectly apply the word "and" when applying set operations.

Instructional Tasks

Instructional Task 1 (MTR.2.1)

Given U is the set of letters of the alphabet, A is the set of vowels and B is the set of letters that come before k, describe the following in words and then write the resulting sets in roster form:

a. A'
b. B'
c. $A \cup B$
d. $A \cap B$
e. $A - B$
f. $B - A$

Instructional Task 2 (MTR.2.1)

Given set A is the set of even numbers less than or equal to six and set B is the set of odd numbers greater than 4 and less than 8, write the following sets in roster form.

a. A
b. B
c. $A \times B$
d. $B \times A$
e. $A - B$
f. $B - A$

Instructional Items

Instructional Item 1

Given $A = \{1,3,5,7,9\}$ and $B = \{1,2,3,4,5\}$, find $A \cap B$.

Instructional Item 2

Given the sets:

$$A = \{2, 5, 8, 9, 11, 13\}$$
$$B = \{1, 2, 3, 4, 5, 9, 10\}$$

Find the following:

- $A \cup B$
- $A \cap B$
- $A \cap B'$

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.LT.5.5

Benchmark

MA.912.LT.5.5 Explore relationships and patterns and make arguments about relationships between sets by using Venn Diagrams.

Connecting Benchmarks/Horizontal Alignment

- MA.912.DP.4.1, MA.912.DP.4.4

Terms from the K-12 Glossary

Vertical Alignment

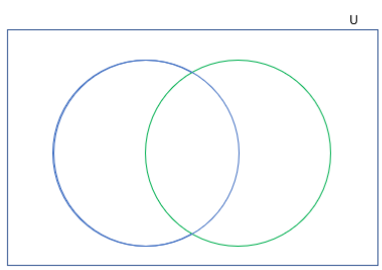
Previous Benchmarks

Next Benchmarks

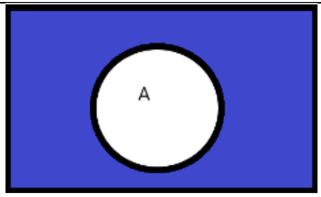
Purpose and Instructional Strategies

In Math for College Liberal Arts students explore relationships between two sets using Venn Diagrams. In other classes students will extend this exploration to include relationships between three or more sets.

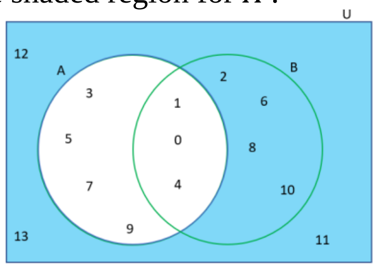
- Instruction includes usage of Venn Diagrams to represent relationships between sets. The universal set U is represented by a rectangle and the sets within the universe are represented by circles.



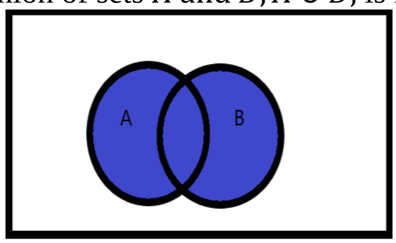
- In a Venn Diagram, the complement, A', is represented by the shaded area.



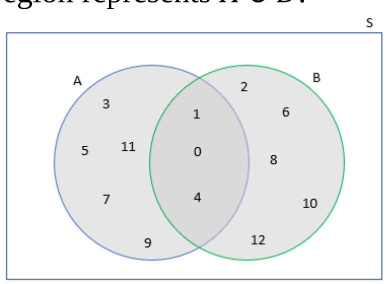
- For example, $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}, A = \{0, 1, 3, 4, 5, 7, 9\}$ and $B = \{0, 1, 2, 4, 6, 8, 10\}$, then $A' = \{2, 6, 8, 10, 11, 12, 13\}$. The Venn Diagram shows the shaded region for A'.



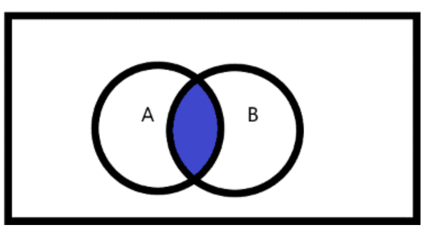
- In a Venn Diagram, the union of sets A and $B, A \cup B$, is represented by the shaded area.



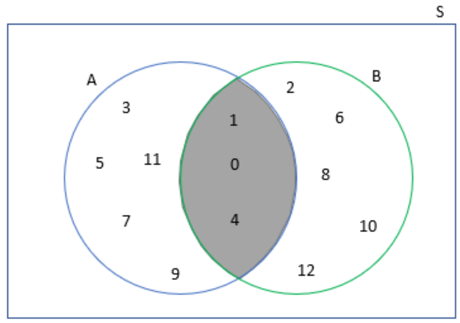
- For example, $A = \{0, 1, 3, 4, 5, 7, 9, 11\}$ and $B = \{0, 1, 2, 4, 6, 8, 10, 12\}$, then $A \cup B$ is $\{0, 1, 3, 4, 5, 7, 9, 11\} \cup \{0, 1, 2, 4, 6, 8, 10, 12\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. The Venn Diagram of this is the following where the shaded region represents $A \cup B$:



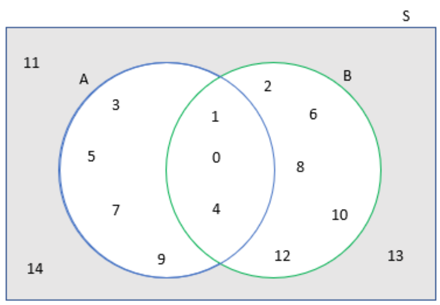
- In a Venn Diagram, the intersection of sets A and B, $A \cap B$, is represented by the shaded area.



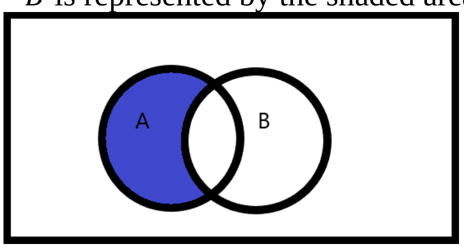
- For example, $A = \{0, 1, 3, 4, 5, 7, 9, 11\}$ and $B = \{0, 1, 2, 4, 6, 8, 10, 12\}$, then $A \cap B$ is $\{0, 1, 3, 4, 5, 7, 9, 11\} \cap \{0, 1, 2, 4, 6, 8, 10, 12\} = \{0, 1, 4\}$. The Venn Diagram of this is the following where the shaded region represents $A \cap B$:



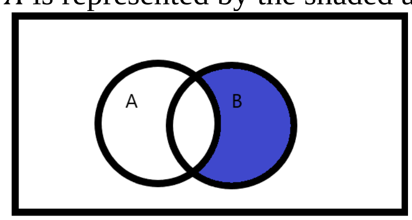
- Operations can be combined, following the order of operations.
    - For example, the Venn Diagram below shows that the shaded region represents the complement of $A \cup B$. $A = \{0, 1, 3, 4, 5, 7, 9\}$ and $B = \{0, 1, 2, 4, 6, 8, 10, 12\}$, then $A \cup B$ is $\{0, 1, 3, 4, 5, 7, 9\} \cup \{0, 1, 2, 4, 6, 8, 10, 12\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12\}$. The $(A \cup B)' = \{11, 13, 14\}$.



- Instruction includes finding the difference of two sets. Order matters when finding the difference of two sets.
- In a Venn Diagram, $A - B$ is represented by the shaded area.



- In a Venn Diagram, $B - A$ is represented by the shaded area.



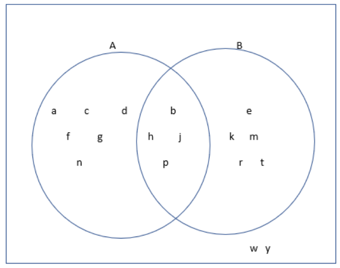
Common Misconceptions or Errors

- Students may repeat elements that are in both sets when writing the union.
- Students may confuse union and intersection.
- Students may incorrectly apply the word "and" when applying set operations.

Instructional Tasks

Instructional Task 1 (MTR.4.1)

Find the following sets using the given Venn Diagram.

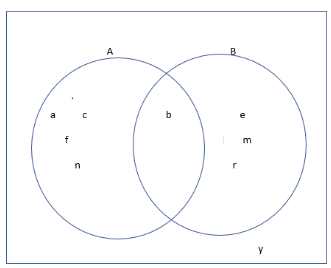


Part A. A
Part B. A'
Part C. B
Part D. B'
Part E. $A \cup B$
Part F. $A \cap B$
Part G. $(A \cap B)'$
Part H. $A - B$
Part I. Describe a set of operations that would result in the set $\{w, y\}$
Part J. Describe a set of operations that would result in the set $\{e, k, m, r, t\}$
Part K. Describe a set of operations that would result in the set
$\{a, b, c, d, f, g, h, j, n, p, w, y\}$

Instructional Items

Instructional Item 1

Find $(A \cup B)'$.



**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.LT.5.6

Benchmark

MA.912.LT.5.6 Prove set relations, including DeMorgan's Laws and equivalence relations.

Connecting Benchmarks/Horizontal Alignment

- MA.912.LT.4.1, MA.912.LT.4.5

Terms from the K-12 Glossary

Vertical Alignment

Previous Benchmarks **Next Benchmarks**

Purpose and Instructional Strategies

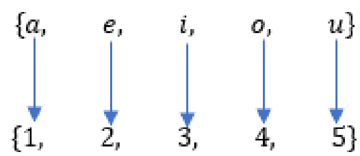
In Math for College Liberal Arts, students begin to prove set relations, starting with DeMorgan's Laws and equivalence relations. In other classes, students will build on this knowledge to include proving other set relations.

- Sets can be described in three ways.
  - Word Description: W is the set of days of the week.
  - Roster Form: elements are listed in { }. The order of the elements does not matter. $W = \{Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday\}$
  - Set Builder Notation: $\{x|x\ is\ \_\_\_\_\_\}$

    This is read "the set of all x such that x is _______"

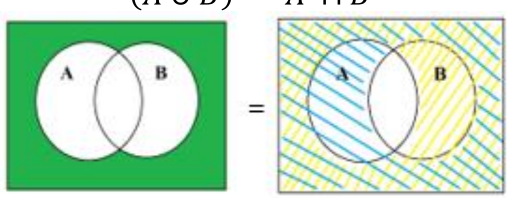
    $$W = \{x|x\ is\ a\ day\ of\ the\ week\}$$
- Instruction includes students brainstorming different sets that have the same number of elements in order to create equivalent relations.
  - For example, students can list two sets with 4 elements each.

    $$Schools = \{Elementary, Middle, K-8, High\}$$
    $$Class = \{Freshman, Sophomore, Junior, Senior\}$$
- Instruction includes proving equivalence relations by showing one-to-one correspondence
  - For example, $\{a, e, i, o, u\}$ is equivalent to $\{1,2,3,4,5\}$.



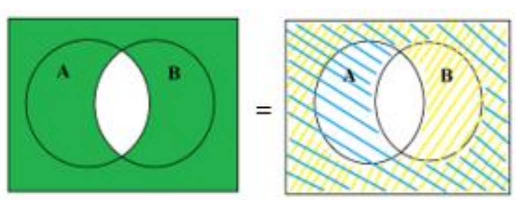
- Instruction includes proving DeMorgan's Laws by drawing Venn Diagrams:

$$(A \cup B)' = A' \cap B'$$



Blue represents the complement of B.
Yellow represents the complement of A.
The Crosshatch represents the intersections of those complements.

$$(A \cap B)' = A' \cup B'$$



Blue represents the complement of B.
Yellow represents the complement of A.
Everything shaded represents the union of those complements.

Common Misconceptions or Errors

- Students may incorrectly try to distribute the negations into $(A \cup B)'$ instead of changing the operator from union to intersection.
- Students may incorrectly try to distribute the negations into $(A \cap B)'$ instead of changing the operator from intersection to union.

Instructional Tasks

Instructional Task 1 (MTR.5.1)

Use Venn Diagrams to prove DeMorgan's Laws.

Instructional Task 2 (MTR.2.1,MTR.4.1)

Part A. Write two equivalent sets in word description.

Part B. Exchange sets with a partner, write the roster forms for each set and show the sets are equivalent.

Instructional Items

Instructional Item 1

Show that the sets $\{green, blue, yellow, red\}$ and $\{square, triangle, circle, hexagon\}$ are equivalent.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*