# Geometry Honors B.E.S.T. Instructional Guide for Mathematics

The B.E.S.T. Instructional Guide for Mathematics (B1G-M) is intended to assist educators with planning for student learning and instruction aligned to Florida's Benchmarks for Excellent Student Thinking (B.E.S.T.) Standards. This guide is designed to aid high-quality instruction through the identification of components that support the learning and teaching of the B.E.S.T. Mathematics Standards and Benchmarks. The B1G-M includes an analysis of information related to the B.E.S.T. Standards for Mathematics within this specific mathematics course, the instructional emphasis and aligned resources. This document is posted on the [B.E.S.T. Standards for Mathematics webpage] of the Florida Department of Education's website and will continue to undergo edits as needed.

## Structural Framework and Intentional Design of the B.E.S.T. Standards for Mathematics

Florida's B.E.S.T. Standards for Mathematics were built on the following.

- The coding scheme for the standards and benchmarks was changed to be consistent with other content areas. The new coding scheme is structured as follows: Content.GradeLevel.Strand.Standard.Benchmark.
- Strands were streamlined to be more consistent throughout.
- The standards and benchmarks were written to be clear and concise to ensure that they are easily understood by all stakeholders.
- The benchmarks were written to allow teachers to meet students' individual skills, knowledge and ability.
- The benchmarks were written to allow students the flexibility to solve problems using a method or strategy that is accurate, generalizable and efficient depending on the content (i.e., the numbers, expressions or equations).
- The benchmarks were written to allow for student discovery (i.e., exploring) of strategies rather than the teaching, naming and assessing of each strategy individually.
- The benchmarks were written to support multiple pathways for success in career and college for students.
- The benchmarks should not be taught in isolation but should be combined purposefully.
- The benchmarks may be addressed at multiple points throughout the year, with the intention of gaining mastery by the end of the year.
- Appropriate progression of content within and across strands was developed for each grade level and across grade levels.
- There is an intentional balance of conceptual understanding and procedural fluency with the application of accurate real-world context intertwined within mathematical concepts for relevance.
- The use of other content areas, like science and the arts, within real-world problems should be accurate, relevant, authentic and reflect grade-level appropriateness.

## Components of the B.E.S.T. Instructional Guide for Mathematics

The following table is an example of the layout for each benchmark and includes the defining attributes for each component. It is important to note that instruction should not be limited to the possible connecting benchmarks, related terms, strategies or examples provided. To do so would strip the intention of an educator meeting students' individual skills, knowledge and abilities.

**Benchmark**
*focal point for instruction within lesson or task*

This section includes the benchmark as identified in the B.E.S.T. Standards for Mathematics. The benchmark, also referred to as the Benchmark of Focus, is the focal point for student learning and instruction. The benchmark, and its related example(s) and clarification(s), can also be found in the course description. The 9-12 benchmarks may be included in multiple courses; select the example(s) or clarification(s) as appropriate for the identified course.

**Connecting Benchmarks/Horizontal Alignment**
*in other standards within the grade level or course*

This section includes a list of connecting benchmarks that relate horizontally to the Benchmark of Focus. Horizontal alignment is the intentional progression of content within a grade level or course linking skills within and across strands. Connecting benchmarks are benchmarks that either make a mathematical connection or include prerequisite skills. The information included in this section is not a comprehensive list, and educators are encouraged to find other connecting benchmarks. Additionally, this list will not include benchmarks from the same standard since benchmarks within the same standard already have an inherent connection.

**Terms from the K-12 Glossary**

This section includes terms from Appendix C: K-12 Glossary, found within the B.E.S.T. Standards for Mathematics document, which are relevant to the identified Benchmark of Focus. The terms included in this section should not be viewed as a comprehensive vocabulary list, but instead should be considered during instruction or act as a reference for educators.

### Vertical Alignment
*across grade levels or courses*

This section includes a list of related benchmarks that connect vertically to the Benchmark of Focus. Vertical alignment is the intentional progression of content from one year to the next, spanning across multiple grade levels. Benchmarks listed in this section make mathematical connections from prior grade levels or courses in future grade levels or courses within and across strands. If the Benchmark of Focus is a new concept or skill, it may not have any previous benchmarks listed. Likewise, if the Benchmark of Focus is a mathematical skill or concept that is finalized in learning and does not have any direct connection to future grade levels or courses, it may not have any future benchmarks listed. The information included in this section is not a comprehensive list, and educators are encouraged to find other benchmarks within a vertical progression.

### Purpose and Instructional Strategies

This section includes further narrative for instruction of the benchmark and vertical alignment. Additionally, this section may also include the following:

- explanations and details for the benchmark;
- vocabulary not provided within Appendix C;
- possible instructional strategies and teaching methods; and
- strategies to embed potentially related Mathematical Thinking and Reasoning Standards (MTRs).

### Common Misconceptions or Errors

This section will include common student misconceptions or errors and may include strategies to address the identified misconception or error. Recognition of these misconceptions and errors enables educators to identify them in the classroom and make efforts to correct the misconception or error. This corrective effort in the classroom can also be a form of formative assessment within instruction.

### Instructional Tasks
*demonstrate the depth of the benchmark and the connection to the related benchmarks*

This section will include example instructional tasks, which may be open-ended and are intended to demonstrate the depth of the benchmark. Some instructional tasks include integration of the Mathematical Thinking and Reasoning Standards (MTRs) and related benchmark(s). Enrichment tasks may be included to make connections to benchmarks in later grade levels or courses. Tasks may require extended time, additional materials and collaboration.

### Instructional Items
*demonstrate the focus of the benchmark*

This section will include example instructional items which may be used as evidence to demonstrate the students' understanding of the benchmark. Items may highlight one or more parts of the benchmark.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

## Mathematical Thinking and Reasoning Standards

*MTRs: Because Math Matters*

Florida students are expected to engage with mathematics through the Mathematical Thinking and Reasoning Standards (MTRs) by utilizing their language as a self-monitoring tool in the classroom, promoting deeper learning and understanding of mathematics. The MTRs are standards which should be used as a lens when planning for student learning and instruction of the B.E.S.T. Standards for Mathematics.

### Structural Framework and Intentional Design of the Mathematical Thinking and Reasoning Standards

The Mathematical Thinking and Reasoning Standards (MTRs) are built on the following.

- The MTRs have the same coding scheme as the standards and benchmarks; however, they are written at the standard level because there are no benchmarks.
- In order to fulfill Florida's unique coding scheme, the 5th place (benchmark) will always be a "1" for the MTRs.
- The B.E.S.T. Standards for Mathematics should be taught through the lens of the MTRs.
- At least one of the MTRs should be authentically and appropriately embedded throughout every lesson based on the expectation of the benchmark(s).
- The bulleted language of the MTRs were written for students to use as self-monitoring tools during daily instruction.
- The clarifications of the MTRs were written for teachers to use as a guide to inform their instructional practices.
- The MTRs ensure that students stay engaged, persevere in tasks, share their thinking, balance conceptual understanding and procedures, assess their solutions, make connections to previous learning and extended knowledge, and apply mathematical concepts to real-world applications.
- The MTRs should not stand alone as a separate focus for instruction, but should be combined purposefully.
- The MTRs will be addressed at multiple points throughout the year, with the intention of gaining mastery of mathematical skills by the end of the year and building upon these skills as they continue in their K-12 education.

## MA.K12.MTR.1.1 Actively participate in effortful learning both individually and collectively.

Mathematicians who participate in effortful learning both individually and with others:

- Analyze the problem in a way that makes sense given the task.
- Ask questions that will help with solving the task.
- Build perseverance by modifying methods as needed while solving a challenging task.
- Stay engaged and maintain a positive mindset when working to solve tasks.
- Help and support each other when attempting a new method or approach.

Clarifications:
Teachers who encourage students to participate actively in effortful learning both individually and with others:

- Cultivate a community of growth mindset learners.
- Foster perseverance in students by choosing tasks that are challenging.
- Develop students' ability to analyze and problem solve.
- Recognize students' effort when solving challenging problems.

## MA.K12.MTR.2.1 Demonstrate understanding by representing problems in multiple ways.

Mathematicians who demonstrate understanding by representing problems in multiple ways:

- Build understanding through modeling and using manipulatives.
- Represent solutions to problems in multiple ways using objects, drawings, tables, graphs and equations.
- Progress from modeling problems with objects and drawings to using algorithms and equations.
- Express connections between concepts and representations.
- Choose a representation based on the given context or purpose.

Clarifications:
Teachers who encourage students to demonstrate understanding by representing problems in multiple ways:

- Help students make connections between concepts and representations.
- Provide opportunities for students to use manipulatives when investigating concepts.
- Guide students from concrete to pictorial to abstract representations as understanding progresses.
- Show students that various representations can have different purposes and can be useful in different situations.

## MA.K12.MTR.3.1 Complete tasks with mathematical fluency.

Mathematicians who complete tasks with mathematical fluency:

- Select efficient and appropriate methods for solving problems within the given context.
- Maintain flexibility and accuracy while performing procedures and mental calculations.
- Complete tasks accurately and with confidence.
- Adapt procedures to apply them to a new context.
- Use feedback to improve efficiency when performing calculations.

Clarifications:
Teachers who encourage students to complete tasks with mathematical fluency:

- Provide students with the flexibility to solve problems by selecting a procedure that allows them to solve efficiently and accurately.
- Offer multiple opportunities for students to practice efficient and generalizable methods.
- Provide opportunities for students to reflect on the method they used and determine if a more efficient method could have been used.

## MA.K12.MTR.4.1 Engage in discussions that reflect on the mathematical thinking of self and others.

Mathematicians who engage in discussions that reflect on the mathematical thinking of self and others:

- Communicate mathematical ideas, vocabulary and methods effectively.
- Analyze the mathematical thinking of others.
- Compare the efficiency of a method to those expressed by others.
- Recognize errors and suggest how to correctly solve the task.
- Justify results by explaining methods and processes.
- Construct possible arguments based on evidence.

Clarifications:
Teachers who encourage students to engage in discussions that reflect on the mathematical thinking of self and others:

- Establish a culture in which students ask questions of the teacher and their peers, and error is an opportunity for learning.
- Create opportunities for students to discuss their thinking with peers.
- Select, sequence and present student work to advance and deepen understanding of correct and increasingly efficient methods.
- Develop students' ability to justify methods and compare their responses to the responses of their peers.

## MA.K12.MTR.5.1 Use patterns and structure to help understand and connect mathematical concepts.

Mathematicians who use patterns and structure to help understand and connect mathematical concepts:

- Focus on relevant details within a problem.
- Create plans and procedures to logically order events, steps or ideas to solve problems.
- Decompose a complex problem into manageable parts.
- Relate previously learned concepts to new concepts.
- Look for similarities among problems.
- Connect solutions of problems to more complicated large-scale situations.

Clarifications:
Teachers who encourage students to use patterns and structure to help understand and connect mathematical concepts:

- Help students recognize the patterns in the world around them and connect these patterns to mathematical concepts.
- Support students to develop generalizations based on the similarities found among problems.
- Provide opportunities for students to create plans and procedures to solve problems.
- Develop students' ability to construct relationships between their current understanding and more sophisticated ways of thinking.

## MA.K12.MTR.6.1 Assess the reasonableness of solutions.

Mathematicians who assess the reasonableness of solutions:

- Estimate to discover possible solutions.
- Use benchmark quantities to determine if a solution makes sense.
- Check calculations when solving problems.
- Verify possible solutions by explaining the methods used.
- Evaluate results based on the given context.

Clarifications:
Teachers who encourage students to assess the reasonableness of solutions:

- Have students estimate or predict solutions prior to solving.
- Prompt students to continually ask, "Does this solution make sense? How do you know?"
- Reinforce that students check their work as they progress within and after a task.
- Strengthen students' ability to verify solutions through justifications.

## MA.K12.MTR.7.1 Apply mathematics to real-world contexts.

Mathematicians who apply mathematics to real-world contexts:

- Connect mathematical concepts to everyday experiences.
- Use models and methods to understand, represent and solve problems.
- Perform investigations to gather data or determine if a method is appropriate.
- Redesign models and methods to improve accuracy or efficiency.

Clarifications:
Teachers who encourage students to apply mathematics to real-world contexts:

- Provide opportunities for students to create models, both concrete and abstract, and perform investigations.
- Challenge students to question the accuracy of their models and methods.
- Support students as they validate conclusions by comparing them to the given situation.
- Indicate how various concepts can be applied to other disciplines.

## Examples of Teacher and Student Moves for the MTRs

Below are examples that demonstrate the embedding of the MTRs within the mathematics classroom. The provided teacher and student moves are examples of how some MTRs could be incorporated into student learning and instruction. The information included in this table is not a comprehensive list, and educators are encouraged to incorporate other teacher and student moves that support the MTRs.

| MTR | Student Moves | Teacher Moves |
|---|---|---|
| MA.K12.MTR.1.1 *Actively participate in effortful learning both individually and collectively.* | • Student asks questions to self, others and teacher when necessary.<br>• Student stays engaged in the task and helps others during the completion of the task.<br>• Student analyzes the task in a way that makes sense to themselves.<br>• Student builds perseverance in self by staying engaged and modifying methods as they solve a problem. | • Teacher builds a classroom community by allowing students to build their own set of "norms."<br>• Teacher creates a culture in which students are encouraged to ask questions, including questioning the accuracy within a real-world context.<br>• Teacher chooses differentiated, challenging tasks that fit the students' needs to help build perseverance in students.<br>• Teacher builds community of learners by encouraging students and recognizing their effort in staying engaged in the task and celebrating errors as an opportunity for learning. |
| MA.K12.MTR.2.1 *Demonstrate understanding by representing problems in multiple ways.* | • Student chooses their preferred method of representation.<br>• Student represents a problem in more than one way and is able to make connections between the representations. | • Teacher plans ahead to allow students to choose their tools.<br>• While sharing student work, teacher purposefully shows various representations to make connections between different strategies or methods.<br>• Teacher helps make connections for students between different representations (i.e., table, equation or written description). |
| MA.K12.MTR.3.1 *Complete tasks with mathematical fluency.* | • Student uses feedback from teacher and peers to improve efficiency. | • Teacher provides opportunity for students to reflect on the method they used, determining if there is a more efficient way depending on the context. |

| MTR | Student Moves | Teacher Moves |
|---|---|---|
| MA.K12.MTR.4.1 *Engage in discussions that reflect on the mathematical thinking of self and others.* | • Student effectively justifies their reasoning for their methods.<br>• Student can identify errors within their own work and create possible explanations.<br>• When working in small groups, student recognizes errors of their peers and offers suggestions.<br>• Student communicates mathematical vocabulary efficiently to others. | • Teacher purposefully groups students together to provide opportunities for discussion.<br>• Teacher chooses sequential representation of methods to help students explain their reasoning. |
| MA.K12.MTR.5.1 *Use patterns and structure to help understand and connect mathematical concepts.* | • Student determines what information is needed and logically follows a plan to solve problems piece by piece.<br>• Student is able to make connections from previous knowledge. | • Teacher allows for students to engage with information to connect current understanding to new methods.<br>• Teacher provides opportunities for students to discuss and develop generalizations about a mathematical concept.<br>• Teacher provides opportunities for students to develop their own steps in solving a problem. |
| MA.K12.MTR.6.1 *Assess the reasonableness of solutions.* | • Student provides explanation of results.<br>• Student continually checks their calculations.<br>• Student estimates a solution before performing calculations. | • Teacher encourages students to check and revise solutions and provide explanations for results.<br>• Teacher allows opportunities for students to verify their solutions by providing justifications to self and others. |
| MA.K12.MTR.7.1 *Apply mathematics to real-world contexts.* | • Student relates their real-world experience to the context provided by the teacher during instruction.<br>• Student performs investigations to determine if a scenario can represent a real-world context. | • Teacher provides real-world context in mathematical problems to support students in making connections using models and investigations. |

## Geometry Honors Areas of Emphasis

In Geometry Honors, instructional time will emphasize five areas:

(1) proving and applying relationships and theorems involving two-dimensional figures using Euclidean geometry and coordinate geometry;
(2) establishing congruence and similarity using criteria from Euclidean geometry and using rigid transformations;
(3) extending knowledge of geometric measurement to two-dimensional figures and three-dimensional figures;
(4) creating and applying equations of circles in the coordinate plane; and
(5) developing an understanding of right triangle trigonometry.

The purpose of the areas of emphasis is not to guide specific units of learning and instruction, but rather provide insight on major mathematical topics that will be covered within this mathematics course. In addition to its purpose, the areas of emphasis are built on the following.

- Supports the intentional horizontal progression within the strands and across the strands in this grade level or course.
- Student learning and instruction should not focus on the stated areas of emphasis as individual units.
- Areas of emphasis are addressed within standards and benchmarks throughout the course so that students are making connections throughout the school year.
- Some benchmarks can be organized within more than one area.
- Supports the communication of the major mathematical topics to all stakeholders.
- Benchmarks within the areas of emphasis should not be taught within the order in which they appear. To do so would strip the progression of mathematical ideas and miss the opportunity to enhance horizontal progressions within the grade level or course.

The table below shows how the benchmarks within this mathematics course are embedded within the areas of emphasis.

| | | Proving & Applying Relationships and Theorems | Congruence and Similarity | Geometric Measurement | Equations of Circles in the Coordinate Plane | Right Triangle Trigonometry |
|---|---|---|---|---|---|---|
| Geometric Reasoning | MA.912.GR.1.1 | x | | x | | x |
| | MA.912.GR.1.2 | x | x | | | |
| | MA.912.GR.1.3 | x | x | x | | x |
| | MA.912.GR.1.4 | x | x | x | | |
| | MA.912.GR.1.5 | x | x | x | | |
| | MA.912.GR.1.6 | x | x | x | | x |
| | MA.912.GR.2.1 | | x | | | |
| | MA.912.GR.2.2 | | x | | | |
| | MA.912.GR.2.3 | | x | | | |
| | MA.912.GR.2.4 | | x | | | |

| | | Proving & Applying Relationships and Theorems | Congruence and Similarity | Geometric Measurement | Equations of Circles in the Coordinate Plane | Right Triangle Trigonometry |
|---|---|---|---|---|---|---|
| | MA.912.GR.2.5 | | x | | | |
| | MA.912.GR.2.6 | | x | | | |
| | MA.912.GR.2.7 | | | | | |
| | MA.912.GR.2.8 | | x | | | |
| | MA.912.GR.2.9 | | | | | |
| | MA.912.GR.3.1 | x | | x | | |
| | MA.912.GR.3.2 | x | | x | x | x |
| | MA.912.GR.3.3 | x | | x | x | x |
| | MA.912.GR.3.4 | | | x | | |
| | MA.912.GR.4.1 | | | x | | |
| | MA.912.GR.4.2 | | | x | | |
| | MA.912.GR.4.3 | | x | x | | |
| | MA.912.GR.4.4 | | | x | | |
| | MA.912.GR.4.5 | | | x | | |
| | MA.912.GR.4.6 | | | x | | |
| | MA.912.GR.5.1 | x | x | | | |
| | MA.912.GR.5.2 | x | x | | | |
| | MA.912.GR.5.3 | x | x | | | |
| | MA.912.GR.5.4 | x | x | | | |
| | MA.912.GR.5.5 | x | x | | | |
| | MA.912.GR.6.1 | x | | | | |
| | MA.912.GR.6.2 | x | | | | |
| | MA.912.GR.6.3 | x | | | | |
| | MA.912.GR.6.4 | x | | | | |
| | MA.912.GR.6.5 | x | x | | x | |
| | MA.912.GR.7.2 | | | | x | |
| | MA.912.GR.7.3 | | | | x | |
| **Trigonometry** | MA.912.T.1.1 | | x | | x | x |
| | MA.912.T.1.2 | | | | | x |
| | MA.912.T.1.3 | | | | | x |
| | MA.912.T.1.4 | | | x | | x |
| **Logic & Discrete Theory** | MA.912.LT.4.3 | x | | | | |
| | MA.912.LT.4.8 | x | | | | |
| | MA.912.LT.4.10 | x | | | | |

**MA.912.GR.1** *Prove and apply geometric theorems to solve problems.*

*MA.912.GR.1.1*

### Benchmark

**MA.912.GR.1.1** Prove relationships and theorems about lines and angles. Solve mathematical and real-world problems involving postulates, relationships and theorems of lines and angles.

Benchmark Clarifications:
*Clarification 1:* Postulates, relationships and theorems include vertical angles are congruent; when a transversal crosses parallel lines, the consecutive angles are supplementary and alternate (interior and exterior) angles and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.
*Clarification 2:* Instruction includes constructing two-column proofs, pictorial proofs, paragraph and narrative proofs, flow chart proofs or informal proofs.
*Clarification 3:* Instruction focuses on helping a student choose a method they can use reliably.

### Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.3.2, MA.912.GR.3.3
- MA.912.GR.5.1, MA.912.GR.5.2
- MA.912.GR.6.1, MA.912.GR.6.2
- MA.912.T.1.1
- MA.912.LT.4.3, MA.912.LT.4.8, MA.912.LT.4.10

### Terms from the K-12 Glossary

- Angle
- Congruent
- Corresponding Angles
- Supplementary Angles
- Transversal
- Vertical

### Vertical Alignment

**Previous Benchmarks**

- MA.3.GR.1.1
- MA.4.GR.1.1
- MA.8.GR.1.4
- MA.912.AR.2.1

**Next Benchmarks**

- MA.912.NSO.3
- MA.912.T.2

### Purpose and Instructional Strategies

In grade 3, students described and identified line segments, rays, perpendicular lines and parallel lines. In grade 4, students classified and solved problems involving acute, right, obtuse, straight and obtuse angle measures. In grade 8, students solved problems involving supplementary, complementary, adjacent and vertical angles. In Geometry, students prove relationships and theorems and solve problems involving lines and angle measure. In later courses, students will study lines and angles in two and three dimensions using vectors and they will use radians to measure angles.

- While focus of this benchmark are those postulates, relationships and theorems listed in *Clarification 1*, instruction includes other definitions, postulates, relationships or theorems such as the midpoint of a segment, angle bisector, segment bisector, perpendicular bisector, the angle addition postulate and the segment addition postulate. Additionally, some postulates and theorems have a converse (i.e., if conclusion, then hypothesis) that can be included.

- Instruction includes the connection to the Logic and Discrete Theory benchmarks when developing proofs. Additionally, with the construction of proofs, instruction reinforces the Properties of Operations, Equality and Inequality. *(MTR.5.1)*
    - For example, when proving that vertical angles are congruent, students must be able to understand and use the Substitution Property of Equality and the Subtraction Property of Equality.
- Instruction utilizes different ways students can organize their reasoning by constructing various proofs when proving geometric statements. It is important to explain the terms statements and reasons, their roles in a geometric proof, and how they must correspond to each other. Regardless of the style, a geometric proof is a carefully written argument that begins with known facts, proceeds from there through a series of logical deductions, and ends with the statement you are trying to prove. *(MTR.2.1)*
    - For examples of different types of proofs, please see MA.912.LT.4.8.
- Instruction includes the connection to compass and straight edge constructions and how the validity of the construction is justified by a proof. *(MTR.5.1)*
- Students should develop an understanding for the difference between a postulate, which is assumed true without a proof, and a theorem, which is a true statement that can be proven. Additionally, students should understand why relationships and theorems can be proven and postulates cannot.
- Instruction includes the use of hatch marks, hash marks, arc marks or tick marks, a form of mathematical notation, to represent segments of equal length or angles of equal measure in diagrams and images.
- Students should understand the difference between congruent and equal. If two segments are congruent (i.e., $\overline{PQ} \cong \overline{MN}$), then they have equivalent lengths (i.e., $PQ = MN$) and the converse is true. If two angles are congruent (i.e., $\angle ABC \cong \angle PQR$), then they have equivalent angle measure (i.e., $m\angle ABC = m\angle PQR$) and the converse is true.
- Instruction includes the use of hands-on manipulatives and geometric software for students to explore relationships, postulates and theorems.
    - For example, folding paper (e.g., patty paper) can be used to explore what happens with angle pairs when two parallel lines are cut by a transversal. Students can discuss the possible pairs of corresponding angles, alternate interior angles, alternate exterior angles and consecutive (same-side interior and same-side exterior) angles. *(MTR.2.1, MTR.4.1)*
- Problem types include mathematical and real-world context where students evaluate the value of a variable that will make two lines parallel; utilize two sets of parallel lines or more than two parallel lines; or write and solve equations to determine an unknown segment length or angle measure.
- Instruction for some relationships or postulates may be necessary in order to prove theorems.
    - For example, to prove that consecutive angles are supplementary when a given transversal crosses parallel lines, students will need to know the postulate that states corresponding angles are congruent.
    - For example, to prove or use the perpendicular bisector theorem, students will need to have knowledge of the definition of a perpendicular bisector of a segment and midpoint.

### Common Misconceptions or Errors

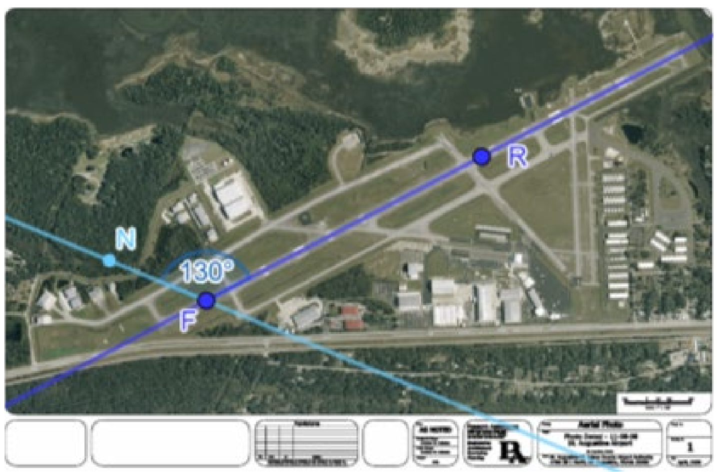
- Students may misuse the terms corresponding, alternate interior and alternate exterior as synonyms of congruent. To help address this conception, students should develop the understanding that these angles are congruent if and only if two parallel lines are cut by a transversal. Similarly, same-side angles are supplementary if and only if the two lines being cut by a transversal are parallel.

## Instructional Tasks

*Instructional Task 1 (MTR.7.1)*

Runways are identified by their orientation relative to Magnetic North as viewed by an approaching aircraft. Runway directions are always rounded to the nearest ten degrees and the zero in the "ones" column is never depicted (i.e., 170 degrees would be viewed as "17" and 20 degrees would be seen as "2"). The same runway has two names which are dependent on the direction of approach.

Use the aerial of Northeast Florida Regional Airport in St. Augustine, FL to answer the questions below.




Part A. Flying to the runway from point $F$, the runway is Runway 13. This means the heading is 130˚ off magnetic north. Draw a line through point $R$ to that goes to magnetic north, what is true about that line and line $NF$?

Part B. On the line drawn in Part A, draw and label a point, $A$.

Part C. Measure angle $FRA$.

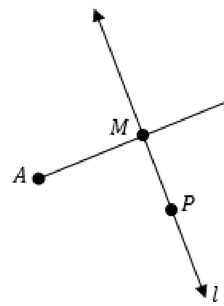
Part D. How does your answer from Part C support you answer from Part A?

Part E. What is the name of the runway when approaching from point $R$?

*Instructional Task 2 (MTR.2.1, MTR.4.1, MTR.5.1)*

Part A. Given $\overline{AB}$, use a compass and straightedge to construct line $l$ such that line $l$ and $\overline{AB}$ form 90° and the point of intersection, $M$, is the midpoint of $\overline{AB}$.

Part B. Suppose that point $P$ lies on line $l$ as shown below. What conjecture can be made about point $P$? Which endpoint of $\overline{AB}$ is closest to point $P$? Use a compass to test your conjecture.




Part C. What if a point, $Q$, was added to line $l$? Which endpoint of $\overline{AB}$ is closest to point $Q$? How does this compare with your conjecture in Part B?

Part D. How can the construction from Part A and the conjectures from Part B and Part C be used to prove that $AP = BP$ given that line $l$ is the perpendicular bisector of $\overline{AB}$ and point $P$ lies on line $l$?

## Instructional Items

*Instructional Item 1*

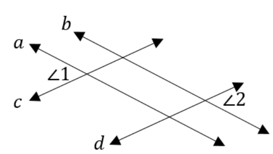
What value of $x$ will make $M$ the midpoint of $\overline{PQ}$ if $PM = 3x - 1$ and $PQ = 5x + 3$?

*Instructional Item 2*

Two lines intersect at point $P$. If the measures of a pair of vertical angles are $(2x - 7)^{\circ}$ and $(x + 13)^{\circ}$, determine $x$ and the measures of the other two angles?

*Instructional Item 3*

Based on the figure below, complete a proof to prove that $\angle 1 \cong \angle 2$ given that $a \parallel b$ and $c \parallel d$.




**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

*MA.912.GR.1.2*

## Benchmark

**MA.912.GR.1.2** Prove triangle congruence or similarity using Side-Side-Side, Side-Angle-Side, Angle-Side-Angle, Angle-Angle-Side, Angle-Angle and Hypotenuse-Leg.

Benchmark Clarifications:
*Clarification 1:* Instruction includes constructing two-column proofs, pictorial proofs, paragraph and narrative proofs, flow chart proofs or informal proofs.
*Clarification 2:* Instruction focuses on helping a student choose a method they can use reliably.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.2.6, MA.912.GR.2.7, MA.912.GR.2.8, MA.912.GR.2.9
- MA.912.GR.5
- MA.912.T.1.1, MA.912.T.1.2, MA.912.T.1.3, MA.912.T.1.4
- MA.912.LT.4.3, MA.912.LT.4.8, MA.912.LT.4.10

## Terms from the K-12 Glossary

- Angle
- Congruent
- Corresponding Angles
- Hypotenuse
- Right Triangle
- Similarity
- Triangle
- Vertical Angles

## Vertical Alignment

**Previous Benchmarks**

- MA.8.GR.1.5
- MA.8.GR.2.1, MA.8.GR.2.2, MA.8.GR.2.4

**Next Benchmarks**

- MA.912.NSO.3
- MA.912.T.1.8

## Purpose and Instructional Strategies

In grade 8, students were introduced to the concepts of congruence and similarity using transformations and proportional relationships. In Geometry, students prove triangle congruence or similarity using Side-Side-Side (SSS), Side-Angle-Side (SAS), Angle-Side-Angle (ASA), Angle-Angle-Side (AAS), Angle-Angle (AA) and Hypotenuse-Leg (HL) criteria. In later courses, these congruence and similarity criteria are critical in understanding applications of vectors and trigonometry.

- Students should develop the understanding of which criteria can be used to prove congruence and which criteria should be used to prove similarity.
  - Criteria that can be used to prove congruence are SSS, SAS, ASA, AAS and HL, where the assumption is that the respective sides in the criterion are equal in length, and the respective angles are equal in measure.
  - Criteria that can be used to prove similarity are AA, SAS and SSS, where the assumption is that the respective sides in the criterion are proportional in length with the same constant of proportionality, and the respective angles are equal in measure.
- Instruction includes developing the understanding that congruence implies similarity. Students should realize that if they have proved congruence of two triangles, then they have also proved similarity of the two triangles.
- Instruction includes the connection to the Logic and Discrete Theory benchmarks when developing proofs. Additionally, with the construction of proofs, instruction reinforces

the Properties of Operations, Equality and Inequality. *(MTR.5.1)*

- Instruction utilizes different ways students can organize their reasoning by constructing various proofs when proving geometric statements. It is important to explain the terms statements and reasons, their roles in a geometric proof, and how they must correspond to each other. Regardless of the style, a geometric proof is a carefully written argument that begins with known facts, proceeds from there through a series of logical deductions, and ends with the statement you are trying to prove. *(MTR.2.1)*
    - For examples of different types of proofs, please see MA.912.LT.4.8.
- Instruction includes the use of hatch marks, hash marks, arc marks or tick marks, a form of mathematical notation, to represent segments of equal length or angles of equal measure in diagrams and images.
- Students should understand the difference between congruent and equal. If two segments are congruent (i.e., $\overline{PQ} \cong \overline{MN}$), then they have equivalent lengths (i.e., $PQ = MN$) and the converse is true. If two angles are congruent (i.e., $\angle ABC \cong \angle PQR$), then they have equivalent angle measure (i.e., $m\angle ABC = m\angle PQR$) and the converse is true.
- Problem types include mathematical or real-world context where students identify which one of the congruence or similarity criteria can be applied in specific cases; deduce information (e.g., vertical angles are congruent, reflexive property for a shared side or angle, corresponding angles when two parallel lines are cut by a transversal are congruent) from given images to determine the congruence or similarity criterion needed to prove triangle congruence or similarity; and determine what piece of information about a pair of triangles must be added to prove triangles are congruent or similar by a certain criterion. *(MTR.6.1)*
- Instruction includes exploring why Hypotenuse-Leg (HL) can be used to show right triangles are congruent. Students should be able to realize that HL is a specific case of Side-Side-Side (SSS) and Side-Angle-Side (SAS) applying the Pythagorean Theorem.
- Instruction for some theorems, relationships or postulates may be necessary in order to prove the validity of congruence or similarity criteria.
    - For example, to prove the validity of the AA similarity criterion, students will need knowledge of the Triangle Sum Theorem.

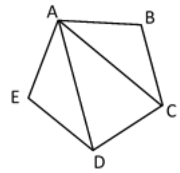
**Common Misconceptions or Errors**

- Students may confuse the congruence and similarity versions of the Side-Side-Side and Side-Angle-Side criteria. To address this misconception, provide students with counterexamples and opportunities to discuss the difference.
- Students may try to use Angle-Angle, Angle-Angle-Angle or Side-Side-Angle to prove congruence.

## Instructional Tasks

*Instructional Task 1 (MTR.4.1, MTR.5.1)*

Pentagon $ABCDE$, as shown below, is a regular pentagon.




Part A. Can you identify two possible congruent triangles in the figure?
Part B. Write a congruence statement for the two triangles that are congruent.
Part C. What theorem or postulate can be used to prove the two triangles congruent?
Part D. Prove that the two triangles chosen in Part A are congruent to one another.
Part E. Determine a triangle that is congruent to triangle $ACD$.
Part F. Repeat Parts B through D with the new pair of triangles.

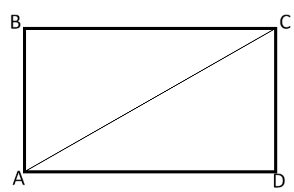
*Instructional Task 2 (MTR.4.1, MTR.5.1)*

Part A. Draw a triangle with side lengths 6 inches, 7 inches and 10 inches. Compare your triangle with a partner.
Part B. Draw a triangle with side lengths 4 inches and 6 inches, and with a 70° angle in between those side lengths. Compare your triangle with a partner.
Part C. Draw a triangle with angle measures of 40° and 60°, and a side length of 5 inches between those angle measures. Compare your triangle with a partner.
Part D. Based on the comparison of triangles created from Parts A, B and C, what can you conclude about criteria for determining triangle congruence?

## Instructional Items

*Instructional Item 1*

Use rectangle $ABCD$ to fill in the blanks.




In a rectangle opposite sides are ________ which means $\overline{\quad} \cong \overline{\quad}$. Triangles $ABC$ and $CDA$ can be proven congruent by Hypotenuse-Leg because ______ is the hypotenuse for both triangles.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

*MA.912.GR.1.3*

## Benchmark

**MA.912.GR.1.3** Prove relationships and theorems about triangles. Solve mathematical and real-world problems involving postulates, relationships and theorems of triangles.

Benchmark Clarifications:
*Clarification 1:* Postulates, relationships and theorems include measures of interior angles of a triangle sum to 180°; measures of a set of exterior angles of a triangle sum to 360°; triangle inequality theorem; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.
*Clarification 2:* Instruction includes constructing two-column proofs, pictorial proofs, paragraph and narrative proofs, flow chart proofs or informal proofs.
*Clarification 3:* Instruction focuses on helping a student choose a method they can use reliably.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.3.1, MA.912.GR.3.2, MA.912.GR.3.3
- MA.912.GR.5.3
- MA.912.T.1.1, MA.912.T.1.2, MA.912.T.1.3, MA.912.T.1.4
- MA.912.LT.4.3, MA.912.LT.4.8, MA.912.LT.4.10

## Terms from the K-12 Glossary

- Angle
- Isosceles Triangle
- Triangle

## Vertical Alignment

**Previous Benchmarks**

- MA.8.GR.1.1, MA.8.GR.1.2, MA.8.GR.1.3
- MA.912.AR.2.1

**Next Benchmarks**

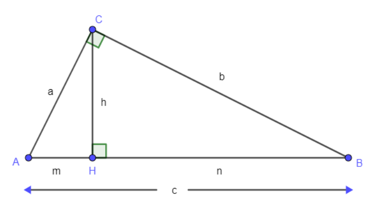
- MA.912.NSO.3
- MA.912.T.1.8

## Purpose and Instructional Strategies

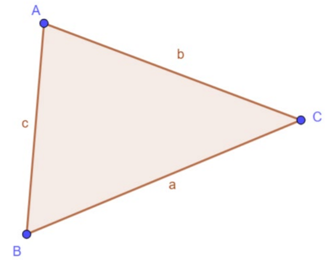
In grade 8, students solved problems involving right triangles, including using the Pythagorean Theorem, and angle measures within triangles. In Geometry, students prove relationships and theorems about triangles and solve problems involving triangles, including right triangles. In later courses, students will use vectors and trigonometry to study and prove further relationships between angle measures and side lengths of triangles.

- While the focus of this benchmark are the postulates, relationships and theorems listed in Clarification 1, instruction could include other definitions, postulates, relationships or theorems like a midsegment of a triangle and other angle or side length measures, the Hinge Theorem and the Scalene Triangle Inequality Theorem. Additionally, instruction includes the converse (i.e., if conclusion, then hypothesis) of some postulates and theorems.
- Instruction includes the connection to the Logic and Discrete Theory benchmarks when developing proofs. Additionally, with the construction of proofs, instruction reinforces the Properties of Operations, Equality and Inequality. *(MTR.5.1)*
- Instruction utilizes different ways students can organize their reasoning by constructing various proofs when proving geometric statements. It is important to explain the terms statements and reasons, their roles in a geometric proof, and how they must correspond to

each other. Regardless of the style, a geometric proof is a carefully written argument that begins with known facts, proceeds from there through a series of logical deductions, and ends with the statement you are trying to prove. *(MTR.2.1)*
    - For examples of different types of proofs, please see MA.912.LT.4.8.
- Instruction includes the connection to compass and straight edge constructions and how the validity of the construction is justified by a proof. *(MTR.5.1)*
- Students should develop an understanding for the difference between a postulate, which is assumed true without a proof, and a theorem, which is a true statement that can be proven. Additionally, students should understand why relationships and theorems can be proven and postulates cannot.
- Instruction includes the use of hatch marks, hash marks, arc marks or tick marks, a form of mathematical notation, to represent segments of equal length or angles of equal measure in diagrams and images.
- Students should understand the difference between congruent and equal. If two segments are congruent (i.e., $\overline{PQ} \cong \overline{MN}$), then they have equivalent lengths (i.e., $PQ = MN$) and the converse is true. If two angles are congruent (i.e., $\angle ABC \cong \angle PQR$), then they have equivalent angle measure (i.e., $m\angle ABC = m\angle PQR$) and the converse is true.
- Instruction includes the use of hands-on manipulatives and geometric software for students to explore relationships, postulates and theorems.
- When proving the Pythagorean Theorem, instruction includes making the connection to similarity criterion.
    - For example, given triangle $ABC$ with $m\angle C = 90^\circ$ and $h$, the height to the hypotenuse, students can begin the proof that $a^2 + b^2 = c^2$ by first proving that $\Delta AHC \sim \Delta CHB \sim \Delta ACB$ using the Angle-Angle (AA) criterion. Students should be able to conclude that, from the definition of similar triangles, $\frac{a}{c} = \frac{m}{a}$ (in $\Delta AHC \sim \Delta ACB$) and $\frac{b}{c} = \frac{n}{b}$ (in $\Delta CHB \sim \Delta ACB$). Students can equivalently rewrite these equations as $a^2 = cm$ and $b^2 = cn$. Then students can add the resulting equations to create $a^2 + b^2 = cm + cn$. Using algebraic reasoning the segment addition postulate, students can conclude that $a^2 + b^2 = c^2$.

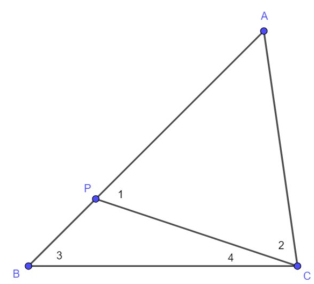



- Instruction includes determining if a triangle can be formed from a given set of sides. The instruction in this benchmark includes discussing the converses of the Triangle Inequality Theorem and the Pythagorean Theorem to understand whether a triangle or a right triangle can be formed from three sides with given side lengths.
    - In triangle $ABC$, with sides $a$, $b$ and $c$, opposite to $A$, $B$ and $C$ respectively, applying the Triangle Inequality Theorem: $a + b > c$, $b + c > a$ and $a + c > b$. From $a + b > c$, $c < a + b$. From $b + c > a$, $c > a - b$ or $a - b < c$. Combining the resulting inequalities: $a - b < c < a + b$.

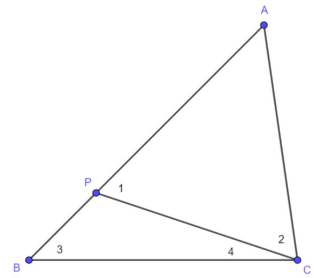



- For example, in a triangle with sides $a$, $b$ and $c$, with $c$ being the longest one, if $a^2 + b^2 = c^2$, then the triangle is right. But if $a^2 + b^2 > c^2$, then the triangle is acute, and if $a^2 + b^2 < c^2$, then the triangle is obtuse.

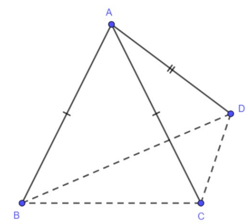
- Instruction includes the understanding of the Scalene Triangle Inequality Theorem (the angle opposite the longer side in a triangle has greater measure), the converse of the Scalene Triangle Inequality Theorem (the side opposite the greater angle in a triangle is longer), and the Hinge Theorem (if two sides of two triangles are congruent and the included angle is different, then the angle that is larger is opposite the longer side).
  - For example, given triangle $ABC$ where $AB > AC$, students can begin the proof of the Scalene Triangle Inequality Theorem by placing the point $P$ such that $AP = AC$. Then, $m\angle 1 = m\angle 2$. As $m\angle ACB > m\angle 2$, $m\angle ACB > m\angle 1$ by the substitution property. $\angle 1$ is exterior to triangle $BPC$, so $m\angle 1 = m\angle 3 + m\angle 4$ and $m\angle 1 > m\angle 3$. Since $m\angle ACB > m\angle 1$ and $m\angle 1 > m\angle 3$, $m\angle ACB > m\angle 3$ or $m\angle ACB > m\angle ABC$. From this we can conclude, if $AB > AC$ then $m\angle ACB > m\angle ABC$.



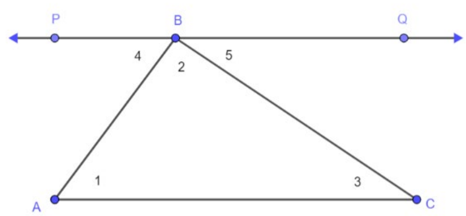

  - For example, given triangle $ABC$ where $m\angle ACB > m\angle ABC$ students can prove the converse of the Scalene Triangle Inequality Theorem by using contradiction to show that $AB > AC$. Students can realize that if $AC > AB$, then the Scalene Triangle Inequality Theorem would imply that $m\angle ABC > m\angle ACB$ contradicting the given information. Additionally, students can realize that if $AB = AC$, then $m\angle ABC = m\angle ABC$, which is another contradiction of the given information. Therefore, students can conclude that $AB > AC$.



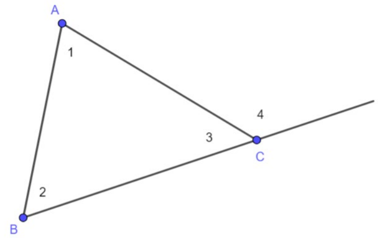

- For example, given triangles $ABD$ and $ACB$ where $AB = AC$ and $m\angle BAD > m\angle CAD$, students can begin the proof of the Hinge Theorem by identifying that triangle $ABC$ is an isosceles triangle with $AB = AC$ and $m\angle ABC = m\angle ACB$. Students should realize that $m\angle DCB > m\angle ACB$, which is the same as $m\angle DCB > m\angle ABC$. Since $m\angle ABC > m\angle DBC$, students can conclude that $m\angle DCB > m\angle DBC$. Applying the Scalene Triangle Inequality Theorem to triangle $BCD$, students can conclude that $DB > DC$.



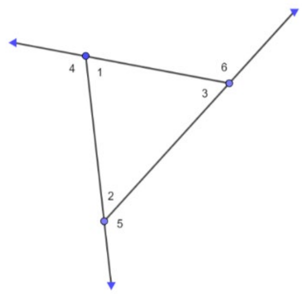

- Instruction includes making the connection to parallel lines and their angle relationships to proving the measures of the interior angles of a triangle sum to 180˚.
  - For example, given triangle $ABC$, students can construct a line through the point $B$ that is parallel $AC$. Students can then explore the relationships between the angle measures in the image below. Students should realize that $m\angle 4 + m\angle 2 + m\angle 5 = 180^\circ$, and $m\angle 1 = m\angle 4$ and $m\angle 3 = m\angle 5$ (Alternate Interior Angles Theorem). Applying the Substitution Property of Equality, students should be able to conclude that $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$.



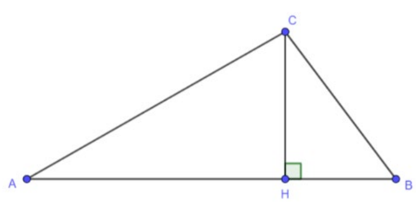

- Instruction includes discussing the relationship between the Triangle Sum Theorem and an exterior angle of a triangle and its two remote interior angles.
  - For example, given triangle $ABC$, students should realize that $m\angle 4 = m\angle 1 + m\angle 2$.




- Instruction for the proof that the measures of a set of exterior angles of a triangle sum to 360° includes the connection to algebraic reasoning skills, the Triangle Sum Theorem and properties of equality.
    - For example, given the triangle below, students should be able to realize that $m\angle 1 + m\angle 4 + m\angle 2 + m\angle 5 + m\angle 3 + m\angle 6 = 540^{\circ}$ since there are three pairs of linear pair angles. Applying properties of equalities and the Triangle Sum Theorem, students should be able to conclude that $m\angle 4 + m\angle 5 + m\angle 6 = 360^{\circ}$.



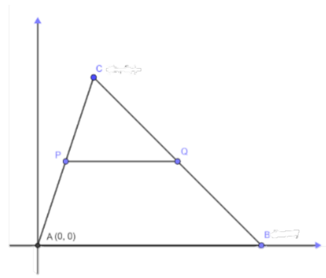

- The proof of the Triangle Inequality Theorem can be approached in a variety of ways. Instruction includes the connection to the Pythagorean Theorem.
    - For example, students can first use the Pythagorean Theorem to prove that the hypotenuse of a right triangle is longer than each of the two legs of the right triangle. Given triangle $ABC$ with an altitude $\overline{CH}$, students can realize that there are two right triangles $ACH$ and $BCH$; with $\overline{AC}$ as the hypotenuse of $\Delta ACH$ and $\overline{CB}$ is the hypotenuse of $\Delta BCH$. Students can use their knowledge of right triangles to determine that $AC > AH$ and $AC > HC$, and $CB > HB$ and $CB > CH$. By adding two of the inequalities, $AC > AH$ and $CB > HB$, students can determine that $AC + CB > AH + HB$ which is equivalent to $AC + CB > AB$ by the Segment Addition Postulate.



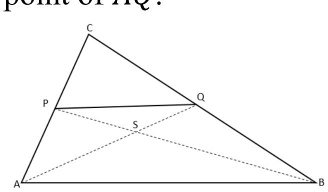

- When proving the Isosceles Triangle Theorem, instruction includes constructing an auxiliary line segment (e.g., median, altitude or angle bisector) from its base to the opposite vertex. *(MTR.2.1)*
    - For example, given triangle $ABC$ with $\overline{AC} \cong \overline{BC}$, student can construct the median from point $C$ to side $AB$, with point of intersection $M$. Students can use the definition of the median of a triangle to state that $\overline{AM} \cong \overline{MC}$. Students should be

able to realize that $\Delta AMC \cong \Delta BMC$ by Side-Side-Side (SSS). So, $\angle A \cong \angle B$ since corresponding parts of congruent triangles are congruent (CPCTC).

- When proving the Triangle Midsegment Theorem, instruction includes the connection to coordinate geometry or to triangle congruence and properties of parallelograms.
    - For example, given triangle $ABC$ on the coordinate plane with $A$ at the origin, $B$ at the point $(b, 0)$ and $C$ at point $(x, y)$. Students can determine the midpoint of $\overline{AC}$ at the point $P$, $\left(\frac{1}{2}x, \frac{1}{2}y\right)$ and the midpoint of $\overline{CB}$ at the point $Q$, $\left(\frac{x+b}{2}, \frac{1}{2}y\right)$. Students should realize that $\overline{PQ}$ is horizontal and is parallel to the base of the triangle, $\overline{AB}$. To determine the length of $PQ$, students can subtract the x-coordinates of $P$ and $Q$ to find it has a length of $\frac{1}{2}b$. Since $PQ = \frac{1}{2}b$ and $AB = b$, then $PQ = \frac{1}{2}AB$ and that $AB = 2(PQ)$.




- Instruction includes the connection between the Triangle Midsegment Theorem and the Trapezoid Midsegment Theorem. *(MTR.5.1)*
    - For example, students can start with a trapezoid and its midsegment then using geometric software, shrink the top base until it has zero length producing a triangle. Students should be able to realize that the average of the lengths of the two bases of the trapezoids becomes one-half the length of the base of the triangle.
- When proving the medians of a triangle meet in a point, instruction includes the connection to coordinate geometry or to the Midpoint Segment Theorem.
    - For example, given triangle $ABC$, students can prove that all medians meet at the same point by first constructing two medians, $AQ$ and $BP$ intersecting at the point $S$. Students should realize that the segment $PQ$ is a midpoint segment. By the Midpoint Segment Theorem, $PQ$ is parallel to $AB$, therefore students can use the Angle-Angle-Angle (AAA) criterion to state that triangles $ASB$ and $QSP$ are similar. Also, by the Midpoint Segment Theorem, $2PQ = AB$. So the scale factor between the two triangles is 2 and students can conclude that $AS = 2QS$ and that point $S$ is the 2:1 partition point of $\overline{AQ}$.




Students can next look at the median from the point $C$ that intersects $\overline{AB}$ at point $R$. Using the same procedure as above, students can prove that median $\overline{CR}$ also intersects the median $\overline{AQ}$ at the 2:1 partition point $S$. Therefore, all three medians

go through the point $S$.
  - For example, students can explore and prove facts about medians of a triangle using the midpoint formula and equations of lines. Students can write the equations of the lines containing two medians and solve the system of equations to determine the point of intersection of the two medians (the centroid). To prove the three medians meet at that point, students can show the centroid is a solution to the equation of the line containing the third median.
  - For example, students can use the notion of the weighted average of two points and its connection to the partitioning of line segments to explore and prove facts about medians. The Centroid Theorem states that the centroid partitions the median from the vertex to the midpoint of the opposite side in the ratio $2:1$. In other words, the centroid is $\frac{2}{3}$ of the way from the vertex to the midpoint of the opposite side. This fact about the centroid can also be applied to show how the medians meet at a point.

## Common Misconceptions or Errors

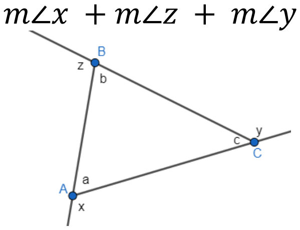
- Students may extend two sides of a triangle when using exterior angles. An exterior angle of a triangle is formed by the extension of one side of the triangle, not two.

## Instructional Tasks

*Instructional Task 1 (MTR.4.1)*

Directions: Print and cut apart the given information, statements and reasons for the proof and provide to students. Students can work individually or in groups. Additionally, students can develop the proof with or without all of the intermediate steps.

Given: $\Delta ABC$ with exterior angles $x$, $y$ and $z$.
Prove: $m\angle x + m\angle z + m\angle y = 360$.




| **Statements** | **Reasons** |
|---|---|
| $\Delta ABC$ with exterior angles $x$, $y$ and $z$ | Given |
| $\angle a$ and $\angle x$ form a linear pair<br>$\angle b$ and $\angle z$ form a linear pair<br>$\angle c$ and $\angle y$ form a linear pair | Definition of linear pair |
| $\angle a$ and $\angle x$ are supplementary<br>$\angle b$ and $\angle z$ are supplementary<br>$\angle c$ and $\angle y$ are supplementary | Angles that form a linear pair are supplementary |
| $m\angle a + m\angle x = 180^\circ$<br>$m\angle b + m\angle z = 180^\circ$<br>$m\angle c + m\angle y = 180^\circ$ | Definition of supplementary angles |
| $m\angle a = 180^\circ - m\angle x$<br>$m\angle b = 180^\circ - m\angle z$<br>$m\angle c = 180^\circ - m\angle y$ | Subtraction property of equality |

| | |
|---|---|
| $m\angle a + m\angle b + m\angle c = 180^\circ$ | Triangle Sum Theorem |
| $180^\circ - m\angle x + 180^\circ - m\angle z + 180^\circ - m\angle y = 180^\circ$ | Substitution property of equality |
| $540^\circ - m\angle x - m\angle z - m\angle y = 180^\circ$ | Commutative and Associative Properties of Addition and Subtraction |
| $-m\angle x - m\angle z - m\angle y = -360^\circ$ | Subtraction property of equality |
| $m\angle x + m\angle z + m\angle y = 360^\circ$ | Multiplication property of equality |

*Instructional Task 2 (MTR.2.1, MTR.4.1, MTR.5.1)*

Provide students with various sizes and types of triangles cut from a paper; large enough for students to tear off the vertices of the triangles. Additionally, provide students tape, glue stick and blank piece of paper.

Part A. Using one of the triangles provided, tear off the vertices.

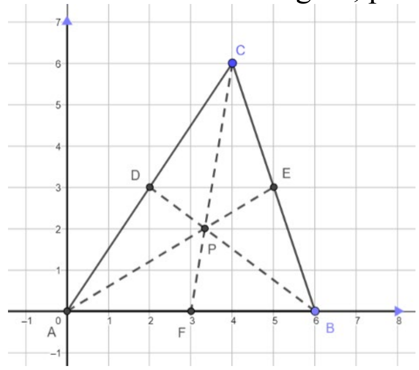
Part B. Place the three vertices in such a way that they are adjacent and create a straight line. If necessary, use tape or glue to keep the vertices in place on the straight line.

Part C. What do you notice about the type of angle the three vertices create? If each of the angle measured are added together, how many degrees does it sum to?

Part D. How does this relate to the Triangle Sum Theorem?

*Instructional Task 3 (MTR.2.1, MTR.4.1, MTR.5.1)*

Given triangle $ABC$ and its medians shown in the figure, prove that they meet in a point, $P$.




Part A. Find the midpoints of the three sides of the triangle $ABC$ ($D$, $E$ and $F$).

Part B. Write the equations of the lines containing two of the medians of the triangle.

Part C. Find the solution of the system of equations created from Part B. Compare your solution with a partner.

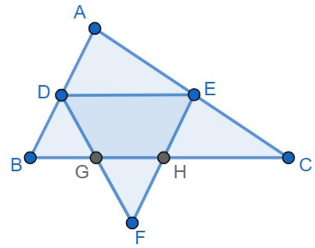
Part D. Write the equation of the line containing the third median of the triangle.

Part E. Check that the solution found from Part C satisfies the equation from Part D. If so, what can you conclude about the three medians of the triangle?

## Instructional Items

*Instructional Item 1*

$\overline{GH}$ is a midsegment of triangle $DEF$ and $\overline{DE}$ is a midsegment of triangle $ABC$. If $GH = 1.5$ cm, what is the length of segment $BC$?




**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

## *MA.912.GR.1.4*

### Benchmark

| | |
|---|---|
| **MA.912.GR.1.4** | Prove relationships and theorems about parallelograms. Solve mathematical and real-world problems involving postulates, relationships and theorems of parallelograms. |

Benchmark Clarifications:
*Clarification 1:* Postulates, relationships and theorems include opposite sides are congruent, consecutive angles are supplementary, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and rectangles are parallelograms with congruent diagonals.
*Clarification 2:* Instruction includes constructing two-column proofs, pictorial proofs, paragraph and narrative proofs, flow chart proofs or informal proofs.
*Clarification 3:* Instruction focuses on helping a student choose a method they can use reliably.

### Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.3.2, MA.912.GR.3.3
- MA.912.LT.4.3, MA.912.LT.4.8, MA.912.LT.4.10

### Terms from the K-12 Glossary

- Angle
- Parallelogram
- Quadrilateral
- Rectangle
- Rhombus
- Square
- Supplementary Angles

### Vertical Alignment

**Previous Benchmarks**

- MA.3.GR.1.2
- MA.5.GR.1.1
- MA.7.GR.1.1
- MA.912.AR.2.1

**Next Benchmarks**

- MA.912.NSO.3

### Purpose and Instructional Strategies

In elementary grades, students identified and classified quadrilaterals, including parallelograms. In grade 7, students solved problems involving the area of parallelograms. In Geometry, students establish relationships between sides, angles and diagonals of parallelograms (including the special cases of parallelograms: rectangles, rhombi and squares), prove the theorems related to these relationships and use them to solve mathematical and real-world problems. In later courses, parallelograms play an important role in work with vectors.

- Relationships, postulates and theorems in this benchmark should focus on, but are not limited to, the ones stated in *Clarification 1*. Additionally, some postulates and theorems have a converse (i.e., if conclusion, then hypothesis) that can be included.
- Instruction includes the connection to the Logic and Discrete Theory benchmarks when developing proofs. Additionally, with the construction of proofs, instruction reinforces the Properties of Operations, Equality and Inequality. *(MTR.5.1)*
- Instruction utilizes different ways students can organize their reasoning by constructing various proofs when proving geometric statements. It is important to explain the terms statements and reasons, their roles in a geometric proof, and how they must correspond to each other. Regardless of the style, a geometric proof is a carefully written argument that begins with known facts, proceeds from there through a series of logical deductions, and ends with the statement you are trying to prove. *(MTR.2.1)*
    - For examples of different types of proofs, please see MA.912.LT.4.8.
- Instruction includes the connection to compass and straight edge constructions and how the validity of the construction is justified by a proof. *(MTR.5.1)*
- Students should develop an understanding for the difference between a postulate, which is assumed true without a proof, and a theorem, which is a true statement that can be proven. Additionally, students should understand why relationships and theorems can be proven and postulates cannot.
- Instruction includes the use of hatch marks, hash marks, arc marks or tick marks, a form of mathematical notation, to represent segments of equal length or angles of equal measure in diagrams and images.
- Students should understand the difference between congruent and equal. If two segments are congruent (i.e., $\overline{PQ} \cong \overline{MN}$), then they have equivalent lengths (i.e., $PQ = MN$) and the converse is true. If two angles are congruent (i.e., $\angle ABC \cong \angle PQR$), then they have equivalent angle measure (i.e., $m\angle ABC = m\angle PQR$) and the converse is true.
- Instruction includes the use of hands-on manipulatives and geometric software for students to explore relationships, postulates and theorems.
- Instruction includes discussing that the definition of a parallelogram only states that the opposite sides are parallel; anything else (e.g., opposite sides are congruent or opposite angles are congruent) is a property and needs to be proven. Students should understand that all parallelograms are trapezoids based on the definition within the K-12 Glossary.
- Instruction includes the connection to triangle congruence and the relationship between angles formed by a transversal through parallel lines when completing proofs about parallelograms.
- When the properties of parallelograms are introduced and proven, it is important to discuss the definitions of rectangles, rhombi and squares. Precision and accuracy are important when discussing the definitions of the special parallelograms. *(MTR.3.1, MTR.4.1)*

- For example, some possible discussion questions include:
  - What makes a parallelogram a rectangle?
  - What is the unique feature of a rhombus?
  - Is a square always a rectangle?
  - Is a square sometimes a rhombus?
  - Is a rectangle always a square? Is a rhombus always a square?
  - What are the properties of a parallelogram observed in a rhombus?
- Instruction includes proving that the diagonals of a parallelogram bisect each other and that the diagonals of a parallelogram are congruent if and only if the parallelogram is a rectangle. Clarify that just having congruent diagonals will not be enough to identify a quadrilateral as a rectangle as this is also a property of isosceles trapezoids. The quadrilateral has to be proven a parallelogram to use this property to classify it as a rectangle.
- Instruction includes the understanding that properties of parallelograms apply to all parallelograms, including squares, rhombi and rectangles.

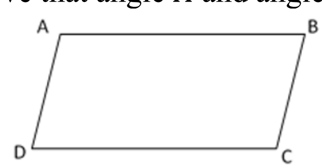
## Common Misconceptions or Errors

- Students may think of squares, parallelograms, rectangles and rhombi as being exclusive to each other. A square is a rectangle and a rhombus.
- Students may think that parallelograms are not trapezoids. The K-12 Mathematics Glossary defines trapezoids as quadrilaterals with at least one pair of parallel sides. Therefore, all parallelograms are trapezoids.

## Instructional Tasks

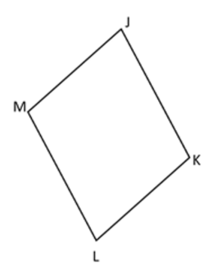
*Instructional Task 1 (MTR.3.1)*

Given parallelogram $ABCD$, prove that angle $A$ and angle $B$ are supplementary.




*Instructional Task 2 (MTR.4.1)*

Given quadrilateral $JKLM$ with $\overline{JK} \cong \overline{LM}$ and $\overline{KL} \cong \overline{MJ}$.




Part A. Draw the diagonal connecting points $M$ and $K$. Determine and prove that two triangles are congruent.

Part B. Using the congruent triangles from Part A, what is true about segments $MJ$ and $LK$?

Part C. Prove that quadrilateral $JKLM$ is a parallelogram.

## Instructional Items

*Instructional Item 1*

Given parallelogram $WXYZ$, where $WX = 2x + 15$, $XY = x + 27$ and $YZ = 4x - 21$, determine the length of $ZW$, in inches.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

### *MA.912.GR.1.5*

#### Benchmark

| | |
|---|---|
| **MA.912.GR.1.5** | Prove relationships and theorems about trapezoids. Solve mathematical and real-world problems involving postulates, relationships and theorems of trapezoids. |

Benchmark Clarifications:

*Clarification 1:* Postulates, relationships and theorems include the Trapezoid Midsegment Theorem and for isosceles trapezoids: base angles are congruent, opposite angles are supplementary and diagonals are congruent.

*Clarification 2:* Instruction includes constructing two-column proofs, pictorial proofs, paragraph and narrative proofs, flow chart proofs or informal proofs.

*Clarification 3:* Instruction focuses on helping a student choose a method they can use reliably.

#### Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.3.2, MA.912.GR.3.3
- MA.912.LT.4.3, MA.912.LT.4.8, MA.912.LT.4.10

#### Terms from the K-12 Glossary

- Angle
- Parallelogram
- Quadrilateral
- Rectangle
- Rhombus
- Square
- Supplementary Angles
- Trapezoid

#### Vertical Alignment

**Previous Benchmarks**

- MA.3.GR.1.2
- MA.5.GR.1.1
- MA.7.GR.1.1
- MA.912.AR.2.1

**Next Benchmarks**

- MA.912.C.4.2

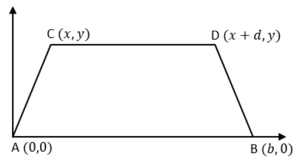
#### Purpose and Instructional Strategies

In elementary grades, students identified and classified trapezoids. In grade 7, students solved problems involving the area of trapezoids. In Geometry, students establish relationships between sides, angles and diagonals of trapezoids (including the special cases of trapezoids: parallelograms, rectangles, rhombi and squares), prove the theorems related to these relationships and use them to solve mathematical and real-world problems. In later courses, students will use trapezoids in cases like determining the area under a curve.

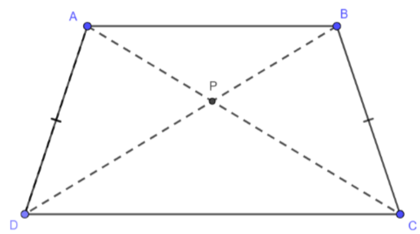
- Relationships, postulates and theorems in this benchmark should focus on, but are not limited to, the ones stated in *Clarification 1*. Additionally, some postulates and theorems have a converse (i.e., if conclusion, then hypothesis) that can be included.
- Instruction includes the connection to the Logic and Discrete Theory benchmarks when

developing proofs. Additionally, with the construction of proofs, instruction reinforces the Properties of Operations, Equality and Inequality. *(MTR.5.1)*

- Instruction utilizes different ways students can organize their reasoning by constructing various proofs when proving geometric statements. It is important to explain the terms statements and reasons, their roles in a geometric proof, and how they must correspond to each other. Regardless of the style, a geometric proof is a carefully written argument that begins with known facts, proceeds from there through a series of logical deductions, and ends with the statement you are trying to prove. *(MTR.2.1)*
    - For examples of different types of proofs, please see MA.912.LT.4.8.
- Instruction includes the connection to compass and straight edge constructions and how the validity of the construction is justified by a proof. *(MTR.5.1)*
- Students should develop an understanding for the difference between a postulate, which is assumed true without a proof, and a theorem, which is a true statement that can be proven. Additionally, students should understand why relationships and theorems can be proven and postulates cannot.
- Instruction includes the use of hatch marks, hash marks, arc marks or tick marks, a form of mathematical notation, to represent segments of equal length or angles of equal measure in diagrams and images.
- Students should understand the difference between congruent and equal. If two segments are congruent (i.e., $\overline{PQ} \cong \overline{MN}$), then they have equivalent lengths (i.e., $PQ = MN$) and the converse is true. If two angles are congruent (i.e., $\angle ABC \cong \angle PQR$), then they have equivalent angle measure (i.e., $m\angle ABC = m\angle PQR$) and the converse is true.
- Instruction includes the use of hands-on manipulatives and geometric software for students to explore relationships, postulates and theorems.
- The K-12 Mathematics Glossary defines trapezoids as a quadrilateral with at least one pair of parallel sides (called the inclusive definition of trapezoid), therefore, all parallelograms are trapezoids.
- The Trapezoid Midsegment Theorem states that a line segment connecting the midpoints of the legs of the trapezoid, called midsegment, is parallel to the bases, and equal to half their sum (or the average of the length of the bases). It is important to introduce and discuss the new vocabulary related to trapezoids: legs (the two non-parallel sides) and bases (the two parallel sides).
- When proving the Trapezoid Midsegment Theorem, instruction includes the connection to coordinate geometry or to triangle congruence and the Triangle Midsegment Theorem.
    - For example, given trapezoid $ACDB$ on the coordinate plane with $A$ at the origin, $B$ at the point $(b, 0)$, $C$ at point $(x, y)$ and D at the point $(x + d, y)$, students can determine the midpoint of $\overline{AC}$ at the point $P$, $\left(\frac{1}{2}x, \frac{1}{2}y\right)$ and the midpoint of $\overline{BD}$ at the point $Q$, $\left(\frac{x+b+d}{2}, \frac{1}{2}y\right)$. Students should realize that $\overline{PQ}$ is horizontal and is parallel to the base of the trapezoid, $\overline{AB}$. To determine the length of $PQ$, students can subtract the $x$-coordinates of $P$ and $Q$ to find it has a length of $\frac{1}{2}(b + d)$. Since $PQ = \frac{1}{2}(b + d)$, $AB = b$ and $CD = d$, then $PQ = \frac{1}{2}(AB + CD)$.




- Instruction includes the understanding that an isosceles trapezoid is a special case of trapezoid where the legs are congruent and base angles are congruent. Properties of isosceles trapezoids include opposite angles are supplementary and diagonals are congruent. When proving properties of isosceles trapezoids, students can use triangle congruence or relationships between angles measures of parallel lines cut by a transversal (*MTR.3.1*).
  - For example, given isosceles trapezoid $ABCD$ and its diagonals, $\overline{AC}$ and $\overline{DB}$, students can prove that the diagonals are congruent, $\overline{AC} \cong \overline{DB}$, by proving that $\Delta ADC \cong \Delta BCD$. Students should be able to realize that they can use Side-Angle-Side (SAS) to prove that $\Delta ADC \cong \Delta BCD$. Then, students can conclude that $\overline{AC} \cong \overline{DB}$.



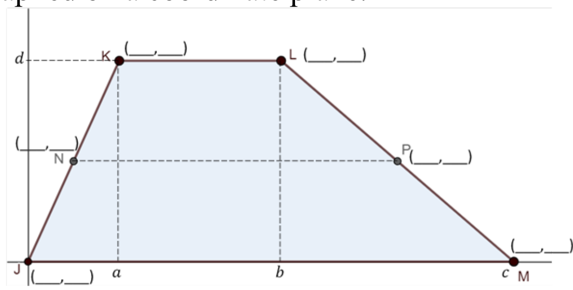

**Common Misconceptions or Errors**

- Students may think that all trapezoids are isosceles trapezoids. To help address this misconception, include examples of different types of trapezoids such as scalene trapezoids, right trapezoids and trapezoids where the parallel sides are not horizontal or vertical.
- Students might think all parallelograms are not trapezoids. To help address this misconception, reiterate the definition of a trapezoid (quadrilateral with at least one pair of parallel sides). Additionally, have students discuss special cases of trapezoids (e.g., isosceles and right) and why they are or are not considered parallelograms.

## Instructional Tasks

*Instructional Task 1 (MTR.3.1)*

Trapezoid $JKLM$ is graphed on a coordinate plane.




Part A. What are the coordinates of points $J$, $K$, $L$ and $M$?

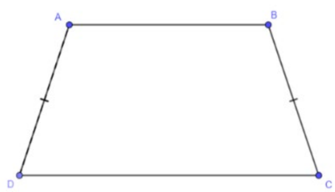
Part B. $N$ is the midpoint of segment $JK$ and $P$ is the midpoint of segment $LM$. What are the coordinates of points $N$ and $P$?

Part C. What are the lengths of segments $KL$, $JK$ and $NP$?

Part D. Use your answers from Parts A through C to prove the Trapezoid Midsegment Theorem.

*Instructional Task 2 (MTR.5.1)*

Isosceles trapezoid $ABCD$ is shown.




Part A. Prove that $\angle A$ is supplementary to $\angle D$ and that $\angle B$ is supplementary to $\angle C$.

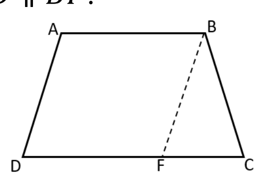
Part B. Prove that $\angle A \cong \angle B$.

Part C. Prove that $\angle A$ is supplementary to $\angle C$ and that $\angle B$ is supplementary to $\angle D$.

Part D. What do you know about isosceles trapezoid $ABCD$ based on the proofs from Parts A to C?

*Instructional Task 3 (MTR.3.1)*

Quadrilateral $ABCD$ is shown with a base that is parallel to its opposite side and has a pair of non-parallel sides. Assume that $\overline{AD} \parallel \overline{BF}$.




Part A. Prove that if non-parallel sides are congruent, then triangle $BFC$ is an isosceles triangle.

Part B. Prove that if the base angles, $\angle C$ and $\angle D$, are congruent, then triangle $BFC$ is an isosceles triangle.

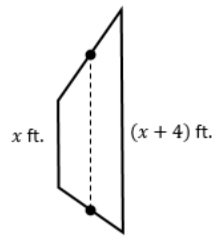
Part C. Prove that the base angles are congruent if and only if the non-parallel opposite sides are congruent.

Part D. Classify the quadrilateral.

## Instructional Items

*Instructional Item 1*

Tulips should be planted three inches apart to give a full look. The Starlings have a trapezoidal plot for a flower garden, as shown in the figure. They are going to put tulips along the parallel sides of the garden. The midsegment to the garden is 10 feet long. Tulips are sold in bags of 25 bulbs.




Part A. What are the lengths of the parallel sides of the garden?
Part B. How many tulips are needed to line the parallel sides?
Part C. What is the minimum number of bags the Starlings need to purchase to have enough bulbs to line the parallel sides of the garden?

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

*MA.912.GR.1.6*

## Benchmark

**MA.912.GR.1.6** Solve mathematical and real-world problems involving congruence or similarity in two-dimensional figures.

Benchmark Clarifications:
*Clarification 1:* Instruction includes demonstrating that two-dimensional figures are congruent or similar based on given information.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.2
- MA.912.GR.3.3
- MA.912.GR.4.3
- MA.912.GR.6.5

## Terms from the K-12 Glossary

- Congruent
- Similarity

## Vertical Alignment

**Previous Benchmarks**
- MA.8.GR.2.4
- MA.912.AR.2.1

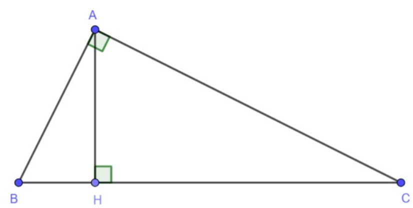
**Next Benchmarks**
- MA.912.T.2

## Purpose and Instructional Strategies

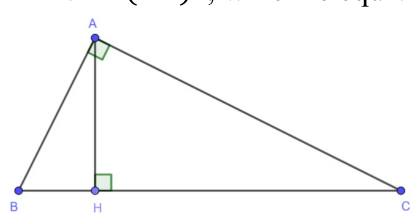
In grade 8 students solved real-world and mathematical problems involving similar triangles, defining similarity in terms of dilations. In Geometry, students extend previous understanding of congruence and similarity to solve mathematical and real-world problems involving congruent or similar polygons. In later courses, students will use similar triangles to develop trigonometry related to the unit circle.

- Instruction includes discussing the definitions of congruent polygons and similar

polygons based on corresponding parts. Students should understand that if a problem involves polygons, they will have to use the definitions of congruent and similar (there are no congruence or similarity criteria for polygons) to show the polygons are congruent or similar and use this information to solve the task. *(MTR.4.1)*
  - When two polygons are congruent, corresponding sides and corresponding angles are congruent.
  - When two polygons are similar, corresponding angles are congruent and corresponding sides are in proportion.
- Problem types includes using congruence and similarity criteria to determine whether two triangles are congruent or similar, and using the definition of congruence and similarity to find missing angle measures and side lengths.
- Instruction includes the understanding of the geometric mean in right triangles and its connection to similarity criteria.
  - Geometric Mean Altitude Theorem
    In a right triangle, the altitude from the right angle to the hypotenuse separates the hypotenuse into two segments. The length of the altitude, in triangle $ABC$, is the geometric mean between the lengths of the two line segments the altitude creates on the hypotenuse. Therefore, by applying the fact that $\Delta ABC \sim \Delta HBA \sim \Delta HAC$, students can conclude that $\frac{HC}{HA} = \frac{HA}{HB}$, which is equivalent to $HC \cdot HB = (HA)^2$, which is equivalent to $\sqrt{HC \cdot HB} = HA$.

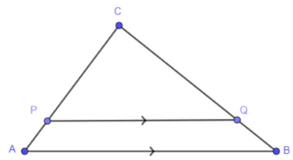



  - Geometric Mean Leg Theorem
    In a right triangle, the altitude from the right angle to the hypotenuse separates the hypotenuse into two segments. The length of one of the legs, in triangle $ABC$, is the geometric mean between the length of the hypotenuse and the line segment of the hypotenuse adjacent to that leg. Therefore, by applying the fact that $\Delta ABC \sim \Delta HBA \sim \Delta HAC$ students can conclude that $\frac{HB}{BA} = \frac{BA}{BC}$, which is equivalent to $HB \cdot BC = (BA)^2$, which is equivalent to $\sqrt{HB \cdot BC} = BA$.




- Instruction includes the connection between triangle similarity and the Triangle Proportionality Theorem, or Side-Splitter Theorem. Students can explore and conclude that if a line is parallel to one side of a triangle intersecting the other two sides of the triangle, then the line divides these two sides proportionally.
  - For example, given triangle $ABC$ and $\overline{PQ} \parallel \overline{AB}$, students can begin the proof

that $\frac{PA}{CP} = \frac{QB}{CQ}$ by first proving $\Delta ABC \sim \Delta PQC$ using the Angle-Angle (AA) criterion. Since corresponding sides of similar triangles are in proportion, students can determine the relationship $\frac{CA}{CP} = \frac{CB}{CQ}$. Students should be able to realize that $CA = CP + PA$ and $CB = CQ + QB$ using the segment addition postulate. Therefore, $\frac{CA}{CP} = \frac{CB}{CQ}$ can be written as $\frac{CP+PA}{CP} = \frac{CQ+QB}{CQ}$ . Students can use their algebraic reasoning to rewrite this relationship equivalently as $\frac{CP}{CP} + \frac{PA}{CP} = \frac{CQ}{CQ} + \frac{QB}{CQ}$, which is equivalent to $1 + \frac{PA}{CP} = 1 + \frac{QB}{CQ}$, which is equivalent to $\frac{PA}{CP} = \frac{QB}{CQ}$.




## Common Misconceptions or Errors

- Students may have difficulty separating overlapping similar or congruent triangles. To help address this misconception, have students draw the two triangles separately with their corresponding known measures and lengths.
- Students may expect that there are congruence or similarity criteria for non-triangular polygons that are like the criteria for triangles. To help address this misconception, have students create examples in which two quadrilaterals are not congruent even though they have corresponding congruent sides to see that there is no Side-Side-Side-Side congruence criterion for quadrilaterals.

## Instructional Tasks

*Instructional Task 1 (MTR.7.1)*

An artist rendering for the Hapbee Honey Company logo is on a 24” x 36” canvas. The company wants to use the logo on a postcard and is determining the size of the logo based on the different mailing costs. According to the United States Postal Service, mailing costs are determined using the following information.

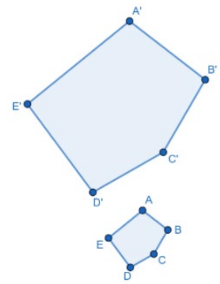
| Postcard Size | Price |
|---|---|
| First-Class Mail® Postcards<br>Maximum size: 6 inches long by 4.25 inches high by 0.016 inch thick | \$0.40 |
| First-Class Mail® Stamped Large Postcards<br>Maximum size: 11.5 inches long by 6.125 inches high by 0.25 inch thick | \$0.58 |

Part A. What is the maximum length of the postcard if it is similar to the original rendering and falls within the First-Class Mail® Postcards dimensions?

Part B. What is the maximum width of the postcard if it is similar to the original rendering and falls within the First-Class Mail® Stamped Large Postcards dimensions?

*Instructional Task 2 (MTR.3.1)*

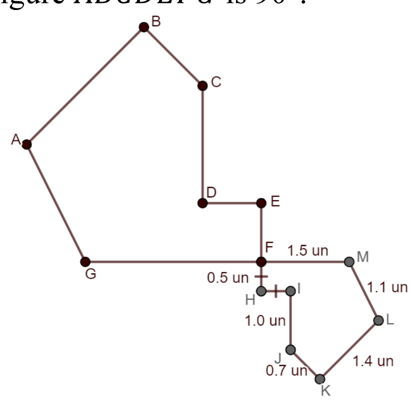
Polygons $ABCDE$ and $A'B'C'D'E'$ are similar and shown.




Part A. If $m\angle A' = 103°$, what is the measure of angle $A$?
Part B. If $m\angle D = 97°$, what other angle has a measure of 97°?
Part C. Find the value of $x$ if $DC = 2x + 1.5$, $D'C' = 5.1$ and $\frac{BC}{B'C'} = \frac{1}{3}$.

*Instructional Task 3 (MTR.5.1)*

Figure $ABCDEFG$ is similar to Figure $LKJIHFM$ with a scale factor of 0.5. Assume that the measure of angle $B$ within Figure $ABCDEFG$ is 90°.




Part A. If point $F$ is located at the origin and line segments $EF$ and $CD$ are vertical and line segments $GF$ and $DE$ are horizontal, determine possible coordinates of each of the points, except points $A$ and $B$, on Figure $ABCDEFG$.
Part B. What is the perimeter of Figure $ABCDEFG$?
Part C. What is the length of segment $DE$?
Part D. What is the perimeter of triangle $AGC$?

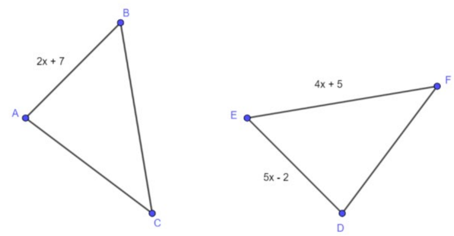
## Instructional Items

*Instructional Item 1*

Triangles $ABC$ and $DEF$ are shown where $\angle A \cong \angle D$, $\angle C \cong \angle F$ and $\overline{AC} \cong \overline{DF}$,

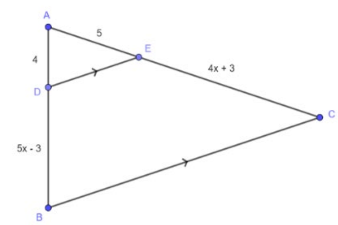
Part A. Determine whether the triangles are congruent.

Part B. If the triangles are congruent, find $EF$, in units.




*Instructional Item 2*

If $\Delta ADE$ and $\Delta ABC$ are similar, what is the length of $\overline{AC}$, in units?




**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

**MA.912.GR.2** *Apply properties of transformations to describe congruence or similarity.*

*MA.912.GR.2.1*

## Benchmark

**MA.912.GR.2.1** Given a preimage and image, describe the transformation and represent the transformation algebraically using coordinates.

*Example:* Given a triangle whose vertices have the coordinates $(-3, 4)$, $(2, 1.7)$ and $(-0.4, -3)$. If this triangle is reflected across the $y$-axis the transformation can be described using coordinates as $(x, y) \rightarrow (-x, y)$ resulting in the image whose vertices have the coordinates $(3, 4)$, $(-2, 1.7)$ and $(0.4, -3)$.

Benchmark Clarifications:

*Clarification 1:* Instruction includes the connection of transformations to functions that take points in the plane as inputs and give other points in the plane as outputs.

*Clarification 2:* Transformations include translations, dilations, rotations and reflections described using words or using coordinates.

*Clarification 3:* Within the Geometry course, rotations are limited to 90°, 180° and 270° counterclockwise or clockwise about the center of rotation, and the centers of rotations and dilations are limited to the origin or a point on the figure.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.3.2, MA.912.GR.3.3
- MA.912.GR.4.3
- MA.912.GR.6.5

## Terms from the K-12 Glossary

- Coordinate Plane
- Dilation
- Origin
- Reflection
- Rigid Transformation
- Rotation
- Scale Factor
- Translation

## Vertical Alignment

**Previous Benchmarks**

- MA.8.GR.2.1
- MA.8.GR.2.2
- MA.8.GR.2.3
- MA.912.F.2.1

**Next Benchmarks**

- MA.912.NSO.4
- MA.912.F.2.2, MA.912.F.2.3, MA.912.F.2.5
- MA.912.T.4

## Purpose and Instructional Strategies

In grade 8, students developed an understanding of single transformations. In Algebra 1, students extended this knowledge of transformations to transforming functions using tables, graphs and equations. In Geometry, students will understand transformations as functions and will transform figures, using two or more transformations, using words and coordinates. In later courses, types of transformations will be expanded to include stretches that can transform functions and conic sections and conversions between rectangular and polar coordinates.

- Instruction includes describing transformations using words and coordinates. Common transformations are provided below.
  - Rotations can be described algebraically as the following:
    90° counterclockwise about the origin $(x, y) \rightarrow (-y, x)$
    90° clockwise about the origin $(x, y) \rightarrow (y, -x)$
    180° counterclockwise about the origin $(x, y) \rightarrow (-x, -y)$
    180° clockwise about the origin $(x, y) \rightarrow (-x, -y)$
    270° counterclockwise about the origin $(x, y) \rightarrow (y, -x)$
    270° clockwise about the origin $(x, y) \rightarrow (-y, x)$
  - Reflections can be described algebraically as the following:
    Over the $x$-axis $(x, y) \rightarrow (x, -y)$
    Over the $y$-axis $(x, y) \rightarrow (-x, y)$
    Over the line $y = x$ $(x, y) \rightarrow (y, x)$
    Over the line $y = -x$ $(x, y) \rightarrow (-y, -x)$
  - Dilations
    Dilation by a factor of $a$, where $a$ is a real number $(x, y) \rightarrow (ax, ay)$
  - Translations
    Horizontal translation by $h$ units, where $h$ is a real number $(x, y) \rightarrow (x + h, y)$
    Vertical translation by $k$ units, where $k$ is a real number $(x, y) \rightarrow (x, y + k)$
    Horizontal translation by $h$ units, where $h$ is a real number, and vertical translation by $k$ units, where $k$ is a real number $(x, y) \rightarrow (x + h, y + k)$

- Instruction includes examining the effect of transforming coordinates by adding, subtracting, or multiplying the $x$- and $y$-coordinates with real-number values to make the connection between functions and transformations.
    - For example, $\Delta PQR$, with vertices $P(-1, 4)$, $Q(3, 4)$ and $R(1, 7)$ can be transformed using the coordinates representation $(x, y) \rightarrow (x, y - 1)$.
    - For example, if the vertices of $\Delta ABC$ are $(4, -2)$, $(4, 5)$ and $(3, 3)$, respectively, and the vertices of $\Delta A'B'C'$ are $(8, -4)$, $(8, 10)$ and $(6, 6)$, respectively, the coordinate representation can be determined.
- Instruction includes the use of hands-on manipulatives and geometric software for students to explore transformations.
- Instruction includes using a variety of ways to describe a transformation using coordinates. *(MTR.2.1)*
    - For example, the same translation can be described using words as 2 units to the right and 4 units down, using coordinates as $(x, y) \rightarrow (x + 2, y - 4)$ or as $T_{x,y} = (x + 2, y - 4)$, or $T_{\langle 2,-4\rangle}$.
    - For example, a reflection over the $x$-axis can be represented as $(x, y) \rightarrow (x, -y)$ or as $r_{x-axis}(x, y) = (x, -y)$.
    - For example, a 90˚ rotation counterclockwise about the origin can be represented as $(x, y) \rightarrow (-y, x)$ or as $R_{O,90^\circ}(x, y) = (-y, x)$ where $O$ is the origin.
- The discussion of translations can be extended to include vectors when describing translations.
    - For example, if a point is translated 3 units to the left and 4 units up the translation vector is $\langle -3,4\rangle$. The vector summarizes the horizontal and vertical shifts.

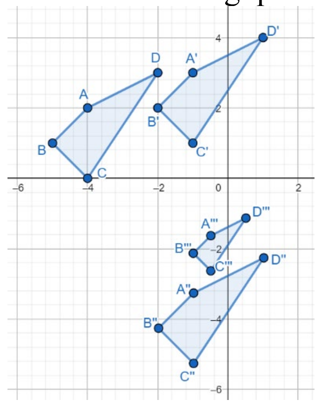
### Common Misconceptions or Errors

- Students may believe the orientation of a figure would be conserved in a rotation in the same way that the orientation of a car, or gondola, is preserved when rotating on a Ferris wheel).
- Student may not be able to visualize some transformations like rotations. To address this, instruction includes using folding paper (e.g., patty paper) or interactive geometric software to allow students hands-on experiences and flexibility in exploration.

## Instructional Tasks

*Instructional Task 1 (MTR.3.1)*

Use the graph to the below to answer the following questions.




Part A. Describe the transformation that maps $ABCD$ to $A'B'C'D'$.

Part B. Represent the transformation described in Part A algebraically.

Part C. Algebraically represent the transformation needed to map $A''B''C''D''$ onto $ABCD$.

Part D. Describe the transformation that maps $A''B''C''D''$ onto $A'''B'''C'''D'''$.

Part E. How is the transformation described in Part D related to the transformation needed to map $A'''B'''C'''D'''$ onto $A''B''C''D''$?

*Instructional Task 2 (MTR.5.1)*

| Preimage | Transformation | Image |
|---|---|---|
| $A(-1.0.5)$ | | $A'(1, 0.5)$ |
| $B(2, 0)$ | ? | $B'(-2, 0)$ |
| $C(3, -3)$ | | $C'(-3, -3)$ |

Part A. Ask students to plot $A$, $B$ and $C$ and $A'$, $B'$ and $C'$on the coordinate plane. What do you notice?

Part B. How can you describe the transformation using words? Explore the patterns among the coordinates of the points of the preimages and the images.

Part C. How can you describe the transformation using coordinate notation?

## Instructional Items

*Instructional Item 1*

A triangle whose vertices are located at $\left(\frac{2}{7}, -1\right)$, $\left(-4, -\frac{14}{5}\right)$ and $(3,1)$ is shifted to the right 2 units.

Part A. What are the coordinates of the triangle after the translation?

Part B. Describe the transformation that would map the preimage to the image algebraically.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

*MA.912.GR.2.2*

## Benchmark

**MA.912.GR.2.2** Identify transformations that do or do not preserve distance.

Benchmark Clarifications:
*Clarification 1:* Transformations include translations, dilations, rotations and reflections described using words or using coordinates.
*Clarification 2:* Instruction includes recognizing that these transformations preserve angle measure.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.1.2, MA.912.GR.1.6
- MA.912.GR.3.2, MA.912.GR.3.3
- MA.912.GR.6.5

## Terms from the K-12 Glossary

- Coordinate Plane
- Dilation
- Origin
- Reflection
- Rigid Transformation
- Rotation
- Scale Factor
- Translation

## Vertical Alignment

**Previous Benchmarks**

- MA.8.GR.2

**Next Benchmarks**

- MA.912.F.2.2, MA.912.F.2.3, MA.912.F.2.5

## Purpose and Instructional Strategies

In grade 8, students were introduced to transformations and whether they preserve congruence or similarity. In Geometry, students determine which transformations preserve distance (rigid transformations, or rigid motions) and which do not (non-rigid transformations, or non-rigid motions), while understanding that all the transformations in this benchmark preserve angle measures. This discussion leads to the definition of congruence and the definition of similarity in terms of rigid and non-rigid motions. *(MTR.3.1, MTR.5.1)* In later courses, other non-rigid motions are studied, including stretches in one coordinate direction.

- Instruction includes the comparison of a variety of geometric figures before and after a single transformation (including reflections, translations, rotations and dilations) to solidify that each of these transformations preserves angle measure. *(MTR.4.1)*
- Instruction includes the student understanding that in order for a transformation based on stretching and shrinking to preserve angle measure, it must have a stretch or shrink of the same scale factor in both the $x$-direction and the $y$-direction.
    - For example, the transformation $(x, y) \rightarrow (2.5x, 2.5y)$ would preserve angle measure, but the transformation $(x, y) \rightarrow (2.5x, 3.5y)$ would not.
- While the intent of the benchmark is to focus on dilations, instruction includes the introduction of other non-rigid transformations to showcase that not all non-rigid transformations preserve angle measure.
    - For example, “stretch” or “shrink” in the direction of one of the axes, such as $(x, y) \rightarrow (5x, y)$ or $(x, y) \rightarrow \left(x, \frac{y}{6}\right)$, does not preserve angle measure. Students can visualize this using geometric software to see how each affects angle

measures of a triangle.

- Transformations should be presented using words and using coordinates.
- Instruction includes the use of folding paper (e.g., patty paper) for hands-on experiences, as needed, and interactive geometry software to allow more flexibility in exploration, when possible.

## Common Misconceptions or Errors

- Students may incorrectly assume that dilations change angle measures.
- Students may assume that dilations only occur in either the $x$-direction or the $y$-direction based on knowledge of function transformations.
- Students may assume that every non-rigid motion preserves angle measures since the only non-rigid motions in the benchmark are dilations, which do preserve angle measures. But many non-rigid motions preserve neither length nor angle measure.

## Instructional Tasks

*Instructional Task 1 (MTR.4.1)*

Penelope made the following statement in Geometry class, "Figure A is a rotation of Figure B about the origin." Chalita disagreed because distance and angle measures are not preserved between the two figures.

- Figure A has the coordinate points $(1, -1)$, $(3, -1)$ and $(1, -2)$.
- Figure B has the coordinate points $(0.8, 1)$, $(1, 3)$ and $(2, 0.8)$.

Part A. What does "distance and angle measures are not preserved" mean in relation to the two figures?

Part B. Determine whether angle measures were preserved from Figure A to Figure B.

Part C. Determine whether distance measures were preserved from Figure A to Figure B.

Part D. Based on your answers from Part B and Part C, determine which student is correct.

*Instructional Task 2 (MTR.3.1)*

Sort the following transformations into preserves distance and does not preserve distance.

| | | |
|---|---|---|
| Counter-clockwise rotation about the origin. | A translation that moves a figure to the right and up. | A reflection over the line $x = 0$. |
| A dilation of $\frac{1}{2}$. | Clockwise rotation about the point $(2, -1)$. | A translation that moves a figure to the left and up. |
| A translation that moves a figure from quadrant I to quadrant III. | A dilation of -3. | Reflection over the $x$-axis. |

## Instructional Items

*Instructional Item 1*

Circle the transformations that can be used when it is important to preserve angle measure.

| | | |
|---|---|---|
| Horizontal Translations | Reflections | Clockwise Rotations |
| Dilations | Vertical Translations | Counterclockwise Rotations |

*Instructional Item 2*

Circle the transformations that can be used when it is important to preserve distance.

| | | |
|---|---|---|
| Horizontal Translations | Reflections | Clockwise Rotations |
| Dilations | Vertical Translations | Counterclockwise Rotations |

*Instructional Item 3*

Write a transformation, or sequence of transformations, that preserves angle measure but does not preserve distance.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

*MA.912.GR.2.3*

## Benchmark

**MA.912.GR.2.3** Identify a sequence of transformations that will map a given figure onto itself or onto another congruent or similar figure.

Benchmark Clarifications:
*Clarification 1:* Transformations include translations, dilations, rotations and reflections described using words or using coordinates.
*Clarification 2:* Within the Geometry course, figures are limited to triangles and quadrilaterals and rotations are limited to 90°, 180°and 270°counterclockwise or clockwise about the center of rotation.
*Clarification 3:* Instruction includes the understanding that when a figure is mapped onto itself using a reflection, it occurs over a line of symmetry.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.1.2
- MA.912.GR.1.6
- MA.912.GR.6.5

## Terms from the K-12 Glossary

- Coordinate Plane
- Dilation
- Origin
- Reflection
- Rigid Transformation
- Rotation
- Scale Factor
- Translation

## Vertical Alignment

**Previous Benchmarks**

- MA.3.GR.1.3
- MA.8.GR.2

**Next Benchmarks**

- MA.912.AR.6.6
- MA.912.T.3.3

## Purpose and Instructional Strategies

In the elementary grades, students learned about lines of symmetry. In grade 8, students learned about the effects of translations, rotations, reflections, and dilations on geometric figures. In Geometry, students use their knowledge of translations, dilations, rotations and reflections to identify a sequence or composition of transformations that map a triangle or a quadrilateral onto another congruent or similar figure or onto itself, and they connect reflections to lines of symmetry. In later courses, lines of symmetry are identified as key features in graphs of polynomials and trigonometric functions.

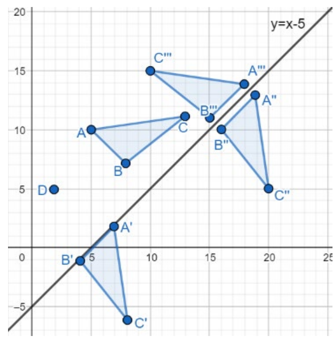
- To describe the sequence of transformations, students will need to know how to describe each one of the transformations in the composition using words or using coordinates. In each case, they will specify vertical and horizontal shifts, center and angle of the rotation, clockwise or counterclockwise, line of reflection, center of the dilation and scale factor, when needed. (*MTR.3.1*)
- Provide multiple opportunities for students to explore mapping a variety of triangles and quadrilaterals onto congruent or similar figures (given the preimage and the image) using both physical exploration (transparencies or patty paper) and virtual exploration when possible. This will allow students to experience multiple compositions of transformations and realize that more than one sequence can be used to map a figure onto another. (*MTR.2.1*)
- Instruction includes examples where preimages and images partially overlap each other.
- When a sequence includes a dilation, it may be helpful that students identify the dilation first, and then continue to identify any rigid motions that may be needed.
- Students can explore the sequence of a reflection over the $x$-axis followed by a reflection over the $y$-axis (or any sequence of two reflections over axes perpendicular to each other). To help students make the connection between different sequences of transformations, ask "Is there a single transformation that produces the same image as this sequence?" (*MTR.5.1*)
- To map a figure onto itself, explore the effect of each transformation. Discuss with students the possibilities of using translations or dilations.
    - When a reflection maps a figure onto itself, the line of reflection is also a line of symmetry for the figure. Explore the lines of symmetries of isosceles and equilateral triangles, and rectangles, rhombi, squares, isosceles trapezoids and kites.
    - When a rotation is used, explore the cases of regular polygons (equilateral triangles and squares) and how to determine the angles of rotation that will map them onto themselves. (*MTR.5.1*)
- Instruction includes discussing the case of a dilation with a scale factor of 1. Even if this case is considered trivial, it leads the conversation to the relationship between congruence and similarity. If a dilation is a similarity transformation, then it produces an image that is similar to the preimage. But if a dilation with a scale factor of 1 produces an image that is congruent to the preimage, then congruence is a case of similarity. In other words, when two figures are congruent, then they are necessarily similar to each other.
- An extension of this benchmark may be to explore the angle of rotation needed to map a regular polygon of 5 or more sides onto itself.

## Common Misconceptions or Errors

- Students may believe there is only one sequence that will lead to the image. Instead, students should explore the fact that multiple sequences will result in the same image.

## Instructional Tasks

*Instructional Task 1 (MTR.2.1)*




Part A. From the list provided, choose and order transformations that could be used to map $\Delta ABC$ onto $\Delta A'''B'''C'''$.

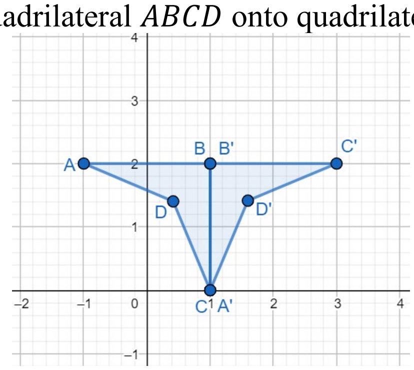
| | |
|---|---|
| Translate vertically 11 units and horizontally 12 units | Rotate 270˚ counterclockwise about point D |
| Rotate 90˚counterclockwise about the origin | Reflect over $y = x$ |
| Reflect over $y = x - 5$ | Translate vertically 12 units and horizontally 11 units |

Part B. Describe the transformation that maps $\Delta A''B''C''$ onto $\Delta A'''B'''C'''$.

## Instructional Items

*Instructional Item 1*

A single rotation mapped quadrilateral $ABCD$ onto quadrilateral $A'B'C'D'$.




Part A. What is the center of the rotation?

Part B. If the rotation is counterclockwise, how many degrees is the rotation?

Part C. Describe another transformation that maps quadrilateral $ABCD$ onto quadrilateral $A'B'C'D'$.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.GR.2.4

## Benchmark

**MA.912.GR.2.4** Determine symmetries of reflection, symmetries of rotation and symmetries of translation of a geometric figure.

<u>Benchmark Clarifications:</u>
*Clarification 1:* Instruction includes determining the order of each symmetry.
*Clarification 2:* Instruction includes the connection between tessellations of the plane and symmetries of translations.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.1.1, MA.912.GR.1.3, MA.912.GR.1.5

## Terms from the K-12 Glossary

- Coordinate Plane
- Line of Symmetry
- Reflection
- Rotation
- Translation

## Vertical Alignment

**Previous Benchmarks**
- MA.3.GR.1.3

**Next Benchmarks**
- MA.912.AR.6.6
- MA.912.T.3

## Purpose and Instructional Strategies

Symmetries of reflection were introduced in the elementary grades through lines of symmetry. In Geometry, students study other types of symmetries coming from rigid transformations that map a polygon onto itself (MA.912.GR.2.1, MA.912.GR.2.3), and they determine the number of times such a transformation must be applied before each point in the polygon is mapped to itself. Symmetries are closely related to features that are used to classify geometric figures. In later courses, students learn that symmetry is a key feature of the graphs of polynomial and trigonometric functions and important to applying such functions in the real world.

- Instruction includes multiple opportunities for students to explore symmetries using both physical exploration (transparencies, mirrors or patty paper) and virtual exploration, when possible.
- Instruction includes using a variety of shapes to explore the reflection symmetry and rotational symmetry of the shapes. Include identifying the lines of symmetry, the order of symmetry and the angle of rotation that will map the figure onto itself.
- The order of symmetry is the smallest (nonzero) number of times that you must apply the corresponding transformation to map each point of the figure onto itself (e.g., 2 for reflections, 4 for a 90˚ rotation).
- The order of a translational symmetry is infinite because no matter how many times one applies it, no point gets mapped onto itself. Translational symmetry results from mapping a figure onto itself by moving it a certain distance in a certain direction. Show students tessellations (covering of a plane using one or more geometric shapes with no overlaps and no gaps), or have them create them, and discuss the translational symmetry in the tessellation. (*MTR.5.1*)

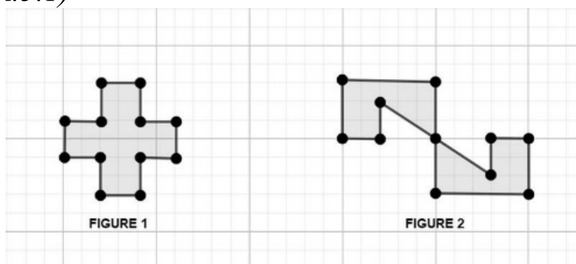
## Common Misconceptions or Errors

- Students may not make the connections to the idea of an infinite pattern (wallpaper pattern, border pattern, etc.) when working with only a portion of that pattern. If students arrive at an order for a translational symmetry that is not infinite, ask them if their answer would be different if the pattern continued.

- Students may have difficulty distinguishing between mapping a figure onto itself and mapping every point of the figure onto itself. To help address this misconception, have students highlight a particular point and observe how it is affected by one application of a transformation that maps the figure onto itself.

## Instructional Tasks

*Instructional Task 1 (MTR.5.1)*




Part A. Draw the lines of symmetry on each figure.

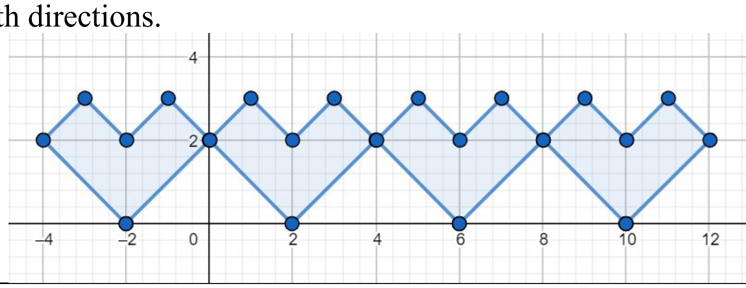
Part B. What is the order of symmetry for reflections and rotations of FIGURE 1? How is each order of symmetry determined?

Part C. Identify the symmetries of FIGURE 2 and determine the order of each symmetry.

## Instructional Items

*Instructional Item 1*

Determine the symmetry of translation for the image below. Assume the pattern continues infinitely in both directions.




**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

*MA.912.GR.2.5*

## Benchmark

**MA.912.GR.2.5** Given a geometric figure and a sequence of transformations, draw the transformed figure on a coordinate plane.

Benchmark Clarifications:
*Clarification 1:* Transformations include translations, dilations, rotations and reflections described using words or using coordinates.
*Clarification 2:* Instruction includes two or more transformations.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.3.2, MA.912.GR.3.3
- MA.912.GR.4.3
- MA.912.GR.6.5

## Terms from the K-12 Glossary

- Coordinate Plane
- Dilation
- Origin
- Reflection
- Rigid Transformation
- Rotation
- Scale Factor
- Translation

## Vertical Alignment

**Previous Benchmarks**

- MA.8.GR.2.1, MA.8.GR.2.2, MA.8.GR.2.3
- MA.912.F.2.1

**Next Benchmarks**

- MA.912.NSO.4
- MA.912.F.2.2, MA.912.F.2.3, MA.912.F.2.5
- MA.912.T.4

## Purpose and Instructional Strategies

In grade 8, students learned about the effects of translations, rotations, reflections, and dilations on geometric figures. In Algebra 1, students extended this knowledge of transformations to transforming functions using tables, graphs and equations. In Geometry, students are given the preimage and a sequence of two or more transformations (a composition) and draw the image on a coordinate plane. In later courses, types of transformations will be expanded to include stretches that can transform functions and conic sections and conversions between rectangular and polar coordinates.

- The purpose of this benchmark is that students produce an image on the coordinate plane as the result of combining transformations (translations, dilations, rotations and reflections).
- Instruction includes transformations described using words and using coordinates.
- Instruction includes understanding that a single transformation can be done as a sequence of transformation. Likewise, some sequences can be described as a single transformation.
    - For example, if the transformation sequence is to translate 1 unit to the left, then rotate 90-degrees clockwise about the origin, then translate 1 unit to the right; the result is the same as rotating the figure 90-degrees clockwise about the point (1,0). This can be described in coordinates as $(x, y) \rightarrow (x-1, y)$, then $(x-1, y) \rightarrow (y, -(x-1))$, then $(y, -(x-1)) \rightarrow (y+1, -(x-1))$, and the result of rotating the figure 90-degrees clockwise about the point $(1,0)$ can be described

in coordinates as $(x, y) \rightarrow (y + 1, -x + 1)$.

- Instruction includes discussing the resulting images using the same preimage and a set of transformations in different orders and understanding that transformations may or not be commutative based on the sequence and type of transformations.
    - For example, if the transformation is the result of a vertical and a horizontal translation, the sequence is commutative.
    - For example, if the transformation is the result of a 180° rotation and a reflection over the $x$-axis, the sequence is commutative.
    - For example, if the transformation is the result of a 90° rotation and a reflection over the $x$-axis, the sequence is not commutative.
- Instruction includes the understanding that if a dilation is part of a sequence, it can be applied at any time within the sequence of transformations without changing the final result.

## Common Misconceptions or Errors

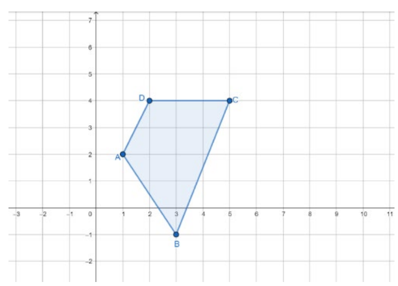
- Students may think order of transformations doesn't matter.

## Instructional Tasks

*Instructional Task 1 (MTR.3.1, MTR.5.1)*

Part A. On the coordinate plane, draw the resulting figure after transforming quadrilateral $ABCD$ through the following sequence below.

- Reflect quadrilateral $ABCD$ over the line $y = x$.
- Translate horizontally and vertically the resulting figure using $(x, y) \rightarrow (x + 3,\ y - 2)$.



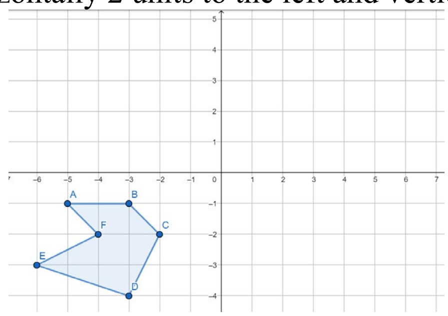

Part B. Would the resulting figure be the same if the transformations were reversed? How did you come to your conclusion?

## Instructional Items

*Instructional Item 1*

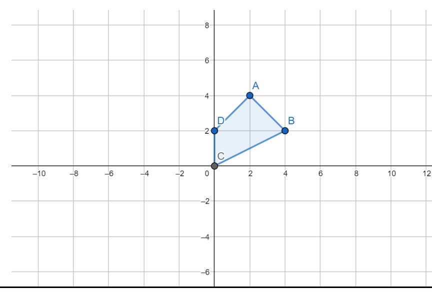
Perform the following sequence of transformations on the polygon $ABCDEF$ on the coordinate plane.

- Rotate 180˚ counterclockwise about the origin.
- Then, translate horizontally 2 units to the left and vertically 3 units down.




*Instructional Item 2*

Draw the resulting figure after quadrilateral $ABCD$ is transformed using $(x, y) \rightarrow (-x, y - 3)$.




**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

*MA.912.GR.2.6*

## Benchmark

**MA.912.GR.2.6** Apply rigid transformations to map one figure onto another to justify that the two figures are congruent.

Benchmark Clarifications:

*Clarification 1:* Instruction includes showing that the corresponding sides and the corresponding angles are congruent.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.1.2
- MA.912.GR.1.6

## Terms from the K-12 Glossary

- Coordinate Plane
- Origin
- Reflection
- Rigid Transformation
- Rotation
- Translation

## Vertical Alignment

| **Previous Benchmarks** | **Next Benchmarks** |
|---|---|
| • MA.8.GR.2.1, MA.8.GR.2.2, MA.8.GR.2.3 | |

## Purpose and Instructional Strategies

In grade 8, students identified a single transformation given the preimage and the image and learned that when the transformation is a reflection, a rotation or a translation, those transformations preserve congruence, that is, the preimage is mapped onto a copy of itself. In Geometry, students determine whether two figures are congruent and justify their answers using a sequence of rigid motions. This leads to the definition of congruence in terms of rigid transformations. (*MTR.5.1*)

- When identifying the transformations in the sequence, specify the vertical and horizontal translations, center and angle of the rotation, clockwise or counterclockwise, line of reflection, when needed.
- Instruction includes describing the rigid transformations using words and using coordinates.
- It is important to identify corresponding parts between the preimage and the image leading to the congruence statement and the congruence of the corresponding parts (angles and sides).
- Instruction includes using examples to compare transformations preserving angle measures and distance versus transformations just preserving angle measures. Include situations where the preimage and the image are not congruent to show how rigid motions will fail mapping one figure onto the other. (*MTR.4.1*)

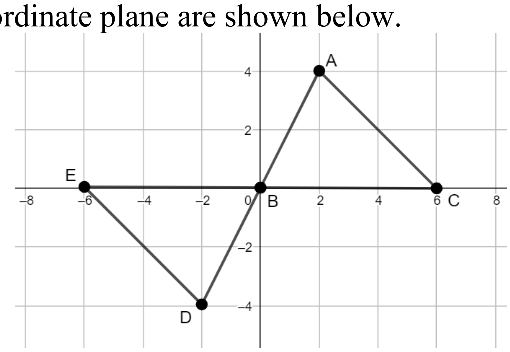
## Common Misconceptions or Errors

- Students may have trouble seeing congruence when a reflection is needed because they limit their thinking to sliding a figure around without turning it over.

## Instructional Tasks

*Instructional Task 1 (MTR.3.1, MTR.4.1)*

Two triangles on the coordinate plane are shown below.



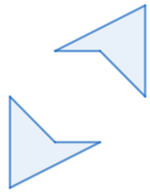

Part A. What transformation(s) could be applied to map triangle $EBD$ onto triangle $CBA$?

Part B. Once the transformation is completed, how can you determine if the two triangles are congruent?

## Instructional Items

*Instructional Item 1*

Describe the sequence of transformations that could be used to prove that the two quadrilaterals shown are congruent.



**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

## *MA.912.GR.2.7*

### Benchmark

**MA.912.GR.2.7** Justify the criteria for triangle congruence using the definition of congruence in terms of rigid transformations.

### Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.1.2
- MA.912.GR.1.6

### Terms from the K-12 Glossary

- Coordinate Plane
- Origin
- Reflection
- Rigid Transformation
- Rotation
- Translation

### Vertical Alignment

**Previous Benchmarks**

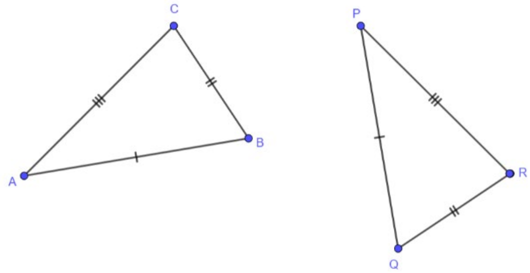
- MA.8.GR.2.1, MA.8.GR.2.2, MA.8.GR.2.3

**Next Benchmarks**

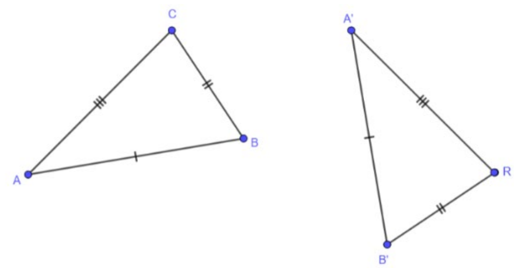
### Purpose and Instructional Strategies

In grade 8, students first learned to associate congruence and similarity with reflections, rotations, translations and dilations. In Geometry, students learn that these transformations provide an equivalent alternative to using congruence and similarity criteria for triangles.

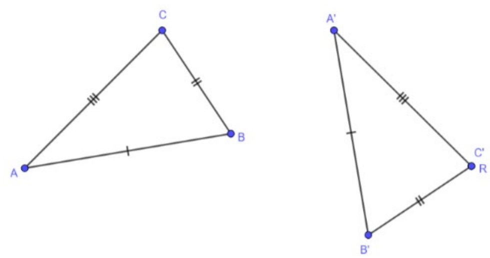
- Instruction includes the connection to the Side-Angle-Side, Angle-Side-Angle (and Angle-Angle-Side), Hypotenuse-Leg and Side-Side-Side congruence criteria. It may be helpful for students to work through one transformation at a time to determine congruence. One approach would be to begin with a translation to map one of the vertices of one triangle to the corresponding vertex of the other. Then, if necessary, a rotation about the first vertex can be used to map a second vertex of one triangle to the corresponding vertex of the other. Finally, it may be necessary to use a reflection to match the corresponding third vertices.
    - For example, triangles $ABC$ and $PQR$ are given below, with $\overline{AB} \cong \overline{PQ}$, $\overline{BC} \cong \overline{QR}$ and $\overline{AC} \cong \overline{PR}$.



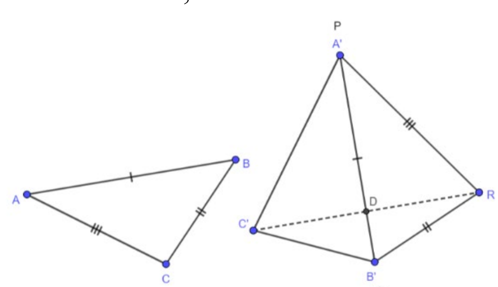

One method to prove the triangles are congruent is as follows: The first step is to discuss the transformations that would map $\overline{AB}$ onto $\overline{PQ}$. Students should be able to identify a convenient sequence of a translation and a rotation, both rigid motions, that will accomplish it. Therefore $\overline{PQ} \cong \overline{A'B'}$ since rigid motions preserve distances.




Applying the same sequence of transformations to point $C$ only, results in $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$ and $\overline{AC} \cong \overline{A'C'}$.




In order to prove that the triangles are congruent, there may be two situations.
Situation A: Point C has already been mapped to point R from the previous sequence. In this case, use the transitive property of congruence, if $\overline{BC} \cong \overline{QR}$ and $\overline{BC} \cong \overline{B'C'}$, then $\overline{QR} \cong \overline{B'C'}$, and if $\overline{AC} \cong \overline{PR}$ and $\overline{AC} \cong \overline{A'C'}$, then $\overline{PR} \cong \overline{A'C'}$ proving triangles $ABC$ and $PQR$ are congruent using transformations.
Situation B: If Point C has not already been mapped to point R from the previous sequence, like shown below, one must use a reflection.




To prove that $C'$ can be mapped onto $R$ using a reflection, use the converse of the

Perpendicular Bisector Theorem. $\overline{A'B'}$ is the perpendicular bisector of $\overline{C'R}$ since $\overline{PR} \cong \overline{A'C'}$ (and $\overline{QR} \cong \overline{B'C'}$). Therefore, $\overline{C'D} \cong \overline{DR}$ and $C'$ can be mapped onto $R$ using a reflection over $\overline{A'B'}$ proving triangles $ABC$ and $PQR$ are congruent using transformations.

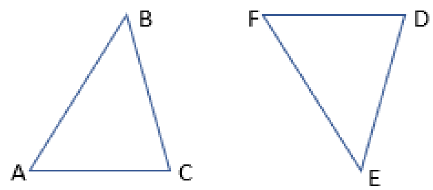
## Common Misconceptions or Errors

- Students may have difficulty understanding the value of having two different, equivalent approaches to proving congruence and similarity. Discuss how approaching a situation in different ways deepens understanding. *(MTR.2.1)*

## Instructional Tasks

*Instructional Task 1 (MTR.2.1, MTR.4.1)*

Triangles $ABC$ and $DEF$ are shown below and $\overline{AB} \cong \overline{EF}$, $\angle B \cong \angle E$ and $\angle A \cong \angle F$.

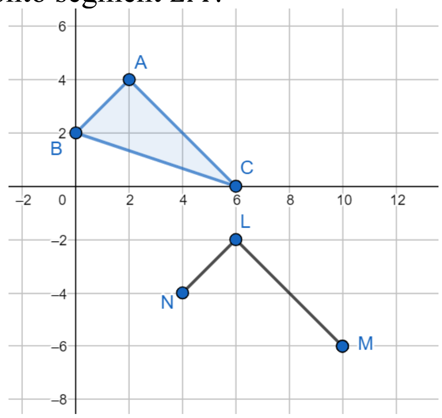



Part A. Describe a sequence of rigid transformations that maps triangle $ABC$ onto triangle $DEF$.

Part B. Compare your sequence with a partner. What do you notice? What information about these triangles makes it possible to determine the rigid transformations in Part A?

*Instructional Task 2 (MTR.4.1)*

When applying the transformation $(x, y) \rightarrow (x + 4, y - 6)$, segment $AB$ maps onto segment $LN$ and segment $AC$ maps onto segment $LM$.



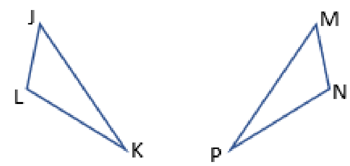

Part A. How can $\Delta ACB \cong \Delta LMN$ be proved using one of the congruence criteria?

Part B. How can $\Delta ACB \cong \Delta LMN$ be proved using rigid transformations?

## Instructional Items

*Instructional Item 1*

Use the image below to complete the sentence.




If $\overline{NP} \cong \boxed{\phantom{x}}$, $\angle P \cong \boxed{\phantom{x}}$ and $\overline{MP} \cong \boxed{\phantom{x}}$, then there is a sequence of rigid transformations that maps $\Delta KJL$ onto $\boxed{\phantom{x}}$.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

*MA.912.GR.2.8*

## Benchmark

**MA.912.GR.2.8** Apply an appropriate transformation to map one figure onto another to justify that the two figures are similar.

Benchmark Clarifications:

*Clarification 1:* Instruction includes showing that the corresponding sides are proportional, and the corresponding angles are congruent.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.1.2, MA.912.GR.1.6
- MA.912.GR.6.5

## Terms from the K-12 Glossary

- Coordinate Plane
- Dilation
- Origin
- Reflection
- Rigid Transformation
- Rotation
- Scale Factor
- Translation

## Vertical Alignment

**Previous Benchmarks**

- MA.8.GR.2.1, MA.8.GR.2.2, MA.8.GR.2.3

**Next Benchmarks**

## Purpose and Instructional Strategies

In grade 8, students identify the scale factor of a dilation given the preimage and the image and learn that when the transformation is a dilation, this transformation does not preserve congruence. The preimage is mapped onto a scaled copy of itself. In Geometry, students determine whether two figures are similar and justify their answers using a dilation (non-rigid motion) or a sequence that includes at least one dilation. This leads to the definition of similarity in terms of dilations and rigid transformations. (*MTR.5.1*)

- When identifying the dilation, specify the center and scale factor of the dilation.
- Instruction includes describing the transformation using words and using coordinates.
- It is important to identify corresponding parts between the preimage and the image leading to the similarity statement and the congruence of the corresponding angles and the proportionality of the corresponding sides.
- Instruction includes using examples to compare transformations. Include situations where the preimage and the image are not similar to show how dilations will fail mapping one figure onto the other. (*MTR.4.1*)
- When proving two triangles are similar, it is important to discuss with the students the effects of choosing which one of the triangles is the pre-image. This affects the scale factor of the dilation.
    - For example, when proving that $\Delta ABC$ and $\Delta PQR$ are similar, the scale factor of the dilation that maps $\Delta ABC$ onto $\Delta PQR$ is $k$, while the scale factor of the dilation that maps $\Delta PQR$ onto $\Delta ABC$ is $\frac{1}{k}$.

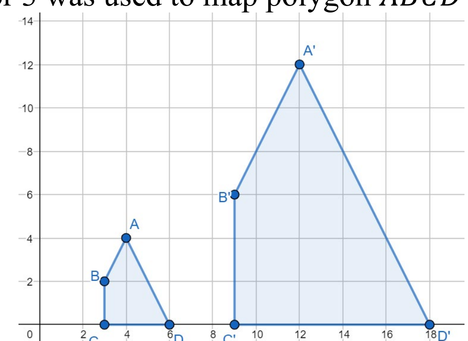
## Common Misconceptions or Errors

- When determining the scale factor of a dilation, students may misidentify the preimage and image, leading to an incorrect scale factor.

## Instructional Tasks

*Instructional Task 1 (MTR.3.1)*

A dilation with scale factor 3 was used to map polygon $ABCD$ onto polygon $A'B'C'D'$.




Part A. Fill in the blanks with either *congruent* or *proportional*.
If the figures are similar, the corresponding sides are ________ and corresponding angles are_________.

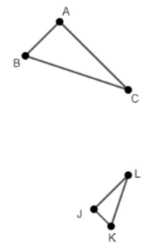
Part B. Identify the sequence of rigid and non-rigid transformations that maps polygon $ABCD$ onto polygon $A'B'C'D'$.

Part B. Use the definition of similarity to prove that polygon $ABCD$ is similar to polygon $A'B'C'D'$. You may need to decompose the polygon into triangles and rectangles.

## Instructional Items

*Instructional Item 1*

In triangles $ABD$ and $JKL$, $m\angle A = m\angle J$, $m\angle C = m\angle L$, and $\overline{AC} = 2\overline{JL}$.




Part A. Describe a sequence of transformations that maps $\Delta ABC$ onto $\Delta JKL$.

Part B. Based on the transformations chosen, determine whether $\Delta ABC$ is congruent or similar to $\Delta JKL$.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

## *MA.912.GR.2.9*

### Benchmark

**MA.912.GR.2.9** Justify the criteria for triangle similarity using the definition of similarity in terms of non-rigid transformations.

### Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.1.2
- MA.912.GR.1.6

### Terms from the K-12 Glossary

- Coordinate Plane
- Dilation
- Origin
- Reflection
- Rigid Transformation
- Rotation
- Scale Factor
- Translation

### Vertical Alignment

**Previous Benchmarks**

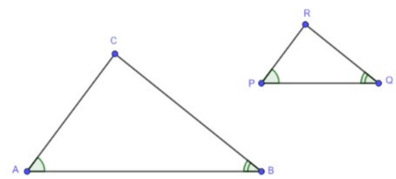
- MA.8.GR.2.1, MA.8.GR.2.2, MA.8.GR.2.3

**Next Benchmarks**

## Purpose and Instructional Strategies

In grade 8, students first learned to associate congruence and similarity with reflections, rotations, translations and dilations. In Geometry, students learn that these transformations provide an equivalent alternative to using congruence and similarity criteria for triangles.

- Instruction includes the connection to the Angle-Angle, Side-Angle-Side, Hypotenuse-Leg and Side-Side-Side similarity criteria and to justifying congruence criteria.
    - For example, if one wants to justify the Angle-Angle criterion, one method is as follows: Start with triangles $ABC$ and $PQR$, with $\angle A \cong \angle P$ and $\angle B \cong \angle Q$, as shown below.




      Students should be able to realize the need of a dilation, in this case with a scale factor $k$ such that $0 < k < 1$ and $k = \frac{PQ}{AB}$. After this dilation, triangle $A'B'C'$ is obtained, such that the length of $A'B'$ equals the length of $PQ$. Since dilations preserve angle measures, $\angle A \cong \angle A'$ and $\angle B \cong \angle B'$. Using the transitive property of congruence, if $\angle A \cong \angle P$ and $\angle A \cong \angle A'$, then $\angle P \cong \angle A'$, and if $\angle B \cong \angle Q$ and $\angle B \cong \angle B'$, then $\angle Q \cong \angle B'$. With $\overline{PQ} \cong \overline{A'B'}$, $\angle P \cong \angle A'$ and $\angle Q \cong \angle B'$, we can prove $\Delta PQR \cong \Delta A'B'C'$ by Side-Angle-Side Congruence Criterion. Additionally, since $\Delta ABC \sim \Delta A'B'C'$ and $\Delta PQR \cong \Delta A'B'C'$, it can be concluded that $\Delta ABC \sim \Delta PQR$. Justification of the other criteria can be done in a similar manner.

## Common Misconceptions or Errors

- When determining the scale factor of a dilation, students may misidentify the preimage and image, leading to an incorrect scale factor.

## Instructional Tasks

*Instructional Task 1 (MTR.3.1, MTR.4.1)*

Triangle $XYZ$ has the coordinates $(0,2)$, $(2,4)$ and $(6,0)$ and triangle $DEF$ has the coordinates $(4,-4)$, $(8,0)$ and $(16,-8)$.

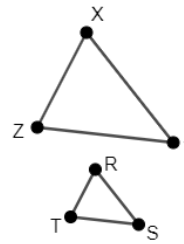
Part A. How can $\Delta ACB \sim \Delta LMN$ be proved using one of the similarity criteria?

Part B. How can $\Delta ACB \sim \Delta LMN$ be proved using rigid and non-rigid transformations?

## Instructional Items

*Instructional Item 1*

Shown below are two triangles where $m\angle X = m\angle R$, $m\angle Y = m\angle S$, and $m\angle Z = m\angle T$. Determine a dilation that maps $\Delta XYZ$ onto $\Delta RST$.




**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

**MA.912.GR.3** *Use coordinate geometry to solve problems or prove relationships.*

*MA.912.GR.3.1*

## Benchmark

**MA.912.GR.3.1** Determine the weighted average of two or more points on a line.

Benchmark Clarifications:
*Clarification 1:* Instruction includes using a number line and determining how changing the weights moves the weighted average of points on the number line.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.1.1
- MA.912.GR.5.2

## Terms from the K-12 Glossary

- Number Line

## Vertical Alignment

**Previous Benchmarks**

- MA.7.AR.3

**Next Benchmarks**

- MA.912.NSO.3
- MA.912.FL.1.3
- MA.912.DP.6.7
- MA.912.LT.2.7
- MA.912.LT.3.3

## Purpose and Instructional Strategies

In grade 7, student developed proportional reasoning skills. In Geometry, students learn the connection between weighted averages and proportional partitioning of line segments. In later course, weighted averages appear in a large number of contexts and they also are used in science courses.

- Instruction focuses on determining the weighted average of two points. This benchmark lays a foundation for weighted averages that will be extended into later courses through multiple pathways. For example, weighted averages can be used to calculate grades (i.e., final grade is 20% on midterm, 50% on final exam and 30% attendance); game theory (i.e., mixed strategies); financial settings (i.e., portfolio consisting of 20% stock and 80% real estate); vectors (i.e., scalar multiplication closely related to partitioning line segments); and probability (i.e., expected value of a random variable is a weighted average of its possible values).

- Instruction includes the connection between weighted averages and partitioning line segments.
    - For example, students will learn that the question "Find the weighted average of the numbers -1 and 5 with weight $\frac{1}{4}$ on the first number and $\frac{3}{4}$ on the second number" is equivalent to the question "What point on the number line is $\frac{3}{4}$ the way from the point -1 to the point 5?"
    - In the prior example, the both questions could be solved by calculating $\frac{1}{4}(-1) + \frac{3}{4}(5)$ which equals 3.5 or by calculuating $-1 + \frac{3}{4}(5 - -1)$ which equals 3.5. Students should be given the flexibility to use either method when solving problems.
- To assist students in their conceptual understanding, or visualization, of weighted averages, instruction includes the use of real weights on yard stick that is balanced on a pivot point. The purpose is not for students to compute the weighted average, but to visual how the weights affect the balance point. Students should explore how the change in weights changes the balance point. It is important to note that in real life, the weight of the yard stick will affect the balance point, if calculated.
    - For example, place three equal weights at 15 inches and 1 weight at 9 inches on the yard stick. If we can neglect the weight of the yard stick, then the balance point will be at 13.5 inches. Since the balance point of the yard stick is at 18 inches, the actual balance point in this experiment will move a little bit towards 18 inches from 13.5 inches, depending on how much the yard stick weighs compared to the weights.
    - For example, a teeter totter with an adult and a child will balance at the pivot point if the adult moves forwards/inward so that the weighted average of the two points is at the pivot point. If the adult weighs 200 pounds and the child weighs 50 pounds, then it is a $4:1$ partition. The weights for the weighted averages can be calculated as $\frac{200}{200+50}$, which equals $\frac{4}{5}$, and as $\frac{50}{200+50}$, which equals $\frac{1}{5}$.

## Common Misconceptions or Errors

- Students may associate the larger weight to the longer segment when visualizing the problem.
- Students may multiply by the weights of the people, or things, instead of multiplying by the weights that lead to the weighted average, which must add up to 1. To help address this misconception, as in the teeter totter example above, students should realize that if they multiply by the weights of the people, then they would need to divide by the sum of the weights of the people.

## Instructional Tasks

*Instructional Task 1 (MTR.2.1, MTR.4.1, MTR.5.1)*

Three numbers are provided below. Use these numbers to answer each question below.

0, 1, 2

Part A. What is the mean ($m_1$) of the three numbers?
Part B. Choose two of the numbers and determine their mean ($m_2$).
Part C. Determine the weighted average of $m_2$ and the third number using the weights $\frac{2}{3}$ and $\frac{1}{3}$. What do you notice?
Part D. Repeat Parts B and C with a different choice of the two numbers.
Part E. Repeat Parts A, B and C with any three real numbers, $x$, $y$ and $z$. Share your answers with a partner. What do you notice?

## Instructional Items

*Instructional Item 1*

What point on the number line is $\frac{7}{9}$ the way from the point -3.6 to the point 10?

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

*MA.912.GR.3.2*

## Benchmark

| | |
|---|---|
| **MA.912.GR.3.2** | Given a mathematical context, use coordinate geometry to classify or justify definitions, properties and theorems involving circles, triangles or quadrilaterals. |

*Example:* Given Triangle $ABC$ has vertices located at $(-2, 2)$, $(3, 3)$ and $(1, -3)$, respectively, classify the type of triangle $ABC$ is based on its angle measures and side lengths.

*Example:* If a square has a diagonal with vertices $(-1, 1)$ and $(-4, -3)$, find the coordinate values of the vertices of the other diagonal and show that the two diagonals are perpendicular.

Benchmark Clarifications:

*Clarification 1:* Instruction includes using the distance or midpoint formulas and knowledge of slope to classify or justify definitions, properties and theorems.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.1.3, MA.912.GR.1.4, MA.912.GR.1.5
- MA.912.GR.7.2

## Terms from the K-12 Glossary

- Circle
- Quadrilateral
- Slope
- Triangle

## Vertical Alignment

**Previous Benchmarks**
- MA.8.AR.3
- MA.8.GR.1.2
- MA.912.AR.2.2
- MA.912.AR.2.3

**Next Benchmarks**
- MA.912.GR.7
- MA.912.T.4

## Purpose and Instructional Strategies

In grade 8 and Algebra 1, students used coordinate systems to study lines and the find distances between points. In Geometry, students expand their knowledge of coordinate geometry to further study lines and distances and relate them to classifying geometric figure. In later courses, coordinates will be used to study a variety of figures, including conic sections and shapes that can be studied using polar coordinates.

- Instruction includes the connection of the Pythagorean Theorem (as was used in grade 8) to the distance formula. It is important that students not depend on just the memorization of the distance formula.
- Instruction includes discussing the convenience of answering with exact values (e.g., the simplest radical form) or with approximations (e.g., rounding to the nearest tenth or hundredth). It is also important to explore the consequences of rounding partial answers on the accuracy or precision of the final answer, especially when working in real-world contexts.
- In this benchmark, instruction is related to circles, triangles and quadrilaterals, their definitions and properties. It may be helpful to review these definitions and properties and the different types of triangles and quadrilaterals as it was part of instruction within elementary grades.
- Instruction includes determining when slope criteria may be necessary.
  - For example, when classifying triangles and quadrilaterals or finding side lengths, the slope criteria may be needed.
  - For example, determining sides of equal measures will decide if a triangle is isosceles or if a quadrilateral is a rhombus.
  - For example, the slope criteria for parallel lines may help when deciding if a quadrilateral is a parallelogram.
  - For example, the slope criteria for perpendicular lines may help when deciding if a triangle is right or if a quadrilateral is a rectangle.
- Explore with the students different approaches for the same goal.
  - For example, given a parallelogram they can determine if it is a rectangle using the slope criteria to identify right angles or using the distance formula (or the Pythagorean Theorem) to identify if the diagonals are congruent.
- Instruction includes opportunities for students to find the coordinates of missing vertices of a triangle or quadrilateral using coordinate geometry and applying definitions, properties, or theorems.
  - For example, when finding the coordinates of $P$ such that $PQRS$ is a rhombus (given the coordinates of $Q$, $R$ and $S$), guide the students to plot the points on the coordinate plane and make a conjecture about the location of $P$. Have students determine if their conjectures are true. Additionally, have students discuss the definitions or properties they may use in each case.

## Common Misconceptions or Errors

- Students may use imprecise methods or incomplete definitions to classify figures.

## Instructional Tasks

*Instructional Task 1 (MTR.2.1, MTR.4.1)*

Part A. What are the coordinates of $P$ if $PQR$ is a right triangle and $Q(-1, 2)$ and $R(3, 0)$?
Part B. Show that $PQ^2 + QR^2 = PR^2$.
Part C. Compare your right triangle with a partner.

*Instructional Task 2 (MTR.3.1)*

Three vertices of quadrilateral $PQRS$ are at the points $Q(-2, 1)$, $R(3, -1)$ and $S(-2, -3)$.
Part A. What are possible coordinates of $P$ if $PQRS$ is a parallelogram?
Part B. Show that $\overline{PR}$ bisects $\overline{QS}$.
Part C. Justify that $PQRS$ is a parallelogram.

*Instructional Task 3 (MTR.3.1, MTR.4.1)*

Coordinates for three two-dimensional figures are given.

Figure A $(2,3)$, $(3, -4)$, $(3, -2)$
Figure B $(3,3)$, $(2, -1)$, $(-2,0)$, $(-1,4)$
Figure C $(-2,3)$, $(-3,1)$, $(0, -4)$, $(3,2)$

Part A. Plot the points on the coordinate plane.
Part B. Write a conjecture about the specific name of each two-dimensional figure. What would you need to determine your conjectures are true?
Part C. Classify each figure.

## Instructional Items

*Instructional Item 1*

Points $A$ $(0,2)$ and $B$ $(2,0)$ are endpoints of segment $AB$, the side of quadrilateral $ABCD$. List possible coordinates for points $C$ and $D$ if quadrilateral $ABCD$ is a rhombus, not a square.

*Instructional Item 2*

Given quadrilateral $ABCD$ with vertices $(-3, -4)$, $(1,5)$, $(5,3)$, and $(5, -8)$, respectively, classify the type of quadrilateral.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

*MA.912.GR.3.3*

## Benchmark

**MA.912.GR.3.3** Use coordinate geometry to solve mathematical and real-world geometric problems involving lines, circles, triangles and quadrilaterals.

*Example:* The line $x + 2y = 10$ is tangent to a circle whose center is located at $(2, -1)$. Find the tangent point and a second tangent point of a line with the same slope as the given line.

*Example:* Given $M(-4, 7)$ and $N(12, -1)$, find the coordinates of point $P$ on $\overline{MN}$ so that $P$ partitions $\overline{MN}$ in the ratio $2: 3$.

Benchmark Clarifications:
*Clarification 1:* Problems involving lines include the coordinates of a point on a line segment including the midpoint.
*Clarification 2:* Problems involving circles include determining points on a given circle and finding tangent lines.
*Clarification 3:* Problems involving triangles include median and centroid.
*Clarification 4:* Problems involving quadrilaterals include using parallel and perpendicular slope criteria.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.1.3, MA.912.GR.1.4, MA.912.GR.1.5
- MA.912.GR.7.2

## Terms from the K-12 Glossary

- Circle
- Diameter
- Quadrilateral
- Radius
- Slope
- Triangle

## Vertical Alignment

**Previous Benchmarks**

- MA.8.AR.3
- MA.8.GR.1.2
- MA.912.AR.2.2, MA.912.AR.2.3
- MA.912.AR.9.1

**Next Benchmarks**

- MA.912.GR.7
- MA.912.T.4

## Purpose and Instructional Strategies

In grade 8 and Algebra 1, students used coordinate systems to study lines and the find distances between points. In Geometry, students expand on their knowledge of coordinate geometry to solve problems geometric problems in real-world and mathematical contexts. In later courses, coordinates will be used to solve a variety of problems involving many shapes, including conic sections and shapes that can be studied using polar coordinates.

- Problem types include finding the midpoint of a segment (midpoint formula); partitioning a segment given endpoints and a ratio; writing the equation of a line, including lines that are parallel or perpendicular; finding the coordinates of the centroid of a triangle; and finding the distance between two points. In some cases, students may need to utilize systems of equations in order to determine solutions.
- Instruction includes the definition of a tangent to a circle and its properties, and the definition of the medians of a triangle and their point of concurrency (centroid).
- Instruction includes various approaches when finding the coordinates of a point partitioning a directed line segment (given the endpoints). *(MTR.2.1, MTR.3.1)* Different

methods are described below.

- The first concept that can be discussed is the connection to weighted average of two points. If the given ratio is $a{:}\,b$, that means the weights of the endpoints $(x_1, y_1)$ and $(x_2, y_2)$ are $\frac{b}{a+b}$ and $\frac{a}{a+b}$, respectively.
  - For example, if the given ratio is $2{:}\,3$ and the points are at $A(-3, 6)$ and $B(4, -8)$ discuss with students where $P$ is on its way from $A$ to $B$. Students should be able to come with $\frac{2}{5}$. That means the weight of $A$ is $\frac{3}{5}$ and the weight of $B$ is $\frac{2}{5}$. The next step is to calculate the $x$-coordinate of $P$ using the weighted averages: $x_P = \frac{3}{5}x_1 + \frac{2}{5}x_2$ which is equivalent to $x_P = \frac{3}{5}(-3) + \frac{2}{5}(4)$ which is equivalent to $x_P = -\frac{1}{5}$. Then calculate the $y$-coordinate of $P$ using the weighted averages: $y_P = \frac{3}{5}y_1 + \frac{2}{5}y_2$ which is equivalent to $y_P = \frac{3}{5}(6) + \frac{2}{5}(-8)$ which is equivalent to $y_P = \frac{2}{5}$. Therefore, $P$ is at $\left(-\frac{1}{5}, \frac{2}{5}\right)$.
- The second method uses the computations of a fraction of the horizontal and the vertical distance between the endpoints (partial distances). That is, if $P$ is partitioning the segment in the ratio $a{:}\,b$ or $\frac{a}{a+b}$ of the way from $(x_1, y_1)$ to $(x_2, y_2)$, then its location is $\frac{a}{a+b}$ of the horizontal distance and $\frac{a}{a+b}$ of the vertical distance from $(x_1, y_1)$ to $(x_2, y_2)$.
  - For example, if the given ratio is $2{:}\,3$ and the points are at $A(-3, 6)$ and $B(4, -8)$ discuss with students where $P$ is on its way from $A$ to $B$. Students should be able to determine the horizontal distance from $A$ to $B$ as $x_2 - x_1 = 4 - (-3)$, which is 7. Then, determine the vertical distance as $y_2 - y_1 = -8 - 6$, which is $-14$. Since the ratio is $2{:}\,3$, $P$ is $\frac{2}{5}$ of the way from $A$ to $B$. Students can calculate $\frac{2}{5}$ of the horizontal and the vertical distance, $\frac{2}{5}(7) = \frac{14}{5}$ and $\frac{2}{5}(-14) = -\frac{28}{5}$, respectively. Students should realize that they will need to add the partial distances to $A$. Therefore, the coordinates of $P$ are $\left(-3 + \frac{14}{5}, 6 - \frac{28}{5}\right) = \left(-\frac{1}{5}, \frac{2}{5}\right)$.
- Other methods include the use of formulas. It is important to note that this would require students to memorize formulas and not encourage students to explore all of the concepts.

- Instruction include that understanding that the midpoint of a segment partitions that segment in the ratio 1:1.
- Instruction includes various approaches when finding medians or centroids of triangles. *(MTR.2.1, MTR.3.1)* Different methods are described below.
  - The equation of the line containing a median can be written from the coordinates of the vertex and the coordinates of the midpoint of the opposite side. Solving a system of equations formed by the equations of the two lines containing two medians of a triangle will result in the coordinates of the centroid.
  - The centroid formula can be given as $Centroid = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$, where

$(x_1, y_1)$, $(x_2, y_2)$ and $(x_3, y_3)$ are the coordinates of the vertices of the triangle. Connections should be made to the fact that the coordinates of the centroid are the means of the coordinates of the vertices.

- The centroid is also the center of gravity of the triangle. Show them with a cardboard triangle that the triangle balances perfectly on its centroid (use a pencil or the tip of your finger).
- A method that makes connections to partitions can be found in the Centroid Theorem. This theorem states that the centroid is $\frac{2}{3}$ of the distance from each vertex to the midpoint of the opposite side, that is, the centroid partitions the median $\frac{2}{3}$ of the way from the vertex to the midpoint of the opposite side. This theorem can also be used to find the coordinates of centroid.
  - For example, given a triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, the midpoints of the sides will be $M(a_1, b_1)$, $N(a_2, b_2)$ and $L(a_3, b_3)$ for $\overline{BC}$, $\overline{AC}$ and $\overline{AB}$, respectively. Applying the Centroid Theorem to the median $\overline{LC}$, the centroid is at $\left(\frac{1}{3}x_3 + \frac{2}{3}a_3, \frac{1}{3}y_3 + \frac{2}{3}b_3\right)$. Using Substitution property of equality, $\left(\frac{1}{3}x_3 + \frac{2}{3}\left(\frac{x_1+x_2}{2}\right), \frac{1}{3}y_3 + \frac{2}{3}\left(\frac{y_1+y_2}{2}\right)\right) = \left(\frac{1}{3}x_3 + \frac{1}{3}x_1 + \frac{1}{3}x_2, \frac{1}{3}y_3 + \frac{1}{3}y_1 + \frac{1}{3}y_2\right)$. This results in $Centroid = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.

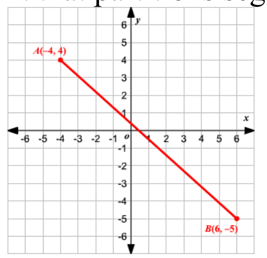
## Common Misconceptions or Errors

- Students may confuse using vertical and horizontal distances when partitioning segments.
- Students may need to be reminded that the segment partition refers to the parts, whose sum is the whole, not a part of a whole.
  - For example, a segment partitioned in a 1:4 ratio is actually 5 parts separated into 1 part and 4 parts, not 1 out of 4 parts.

## Instructional Tasks

*Instructional Task 1 (MTR.3.1)*

What are the coordinates of the point that partitions segment $AB$ in the ratio $2:3$?




*Instructional Task 2 (MTR.5.1)*

Circle $A$ has center located at $(2, 2)$ and contains the point $(4,4)$.

Part A. Write the equation that describes circle $A$.

Part B. Write the equation of a line tangent to Circle $A$ at $(4,4)$.

Part C. Find the equation of a vertical tangent line and of a horizontal tangent line.

*Instructional Task 3 (MTR.3.1)*

Triangle $ABC$ has two of its the vertices located at $(-4, -1)$ and $(3, -3)$.

Part A. Triangle $ABC$ has a centroid located at $\left(-1, \frac{1}{3}\right)$. What is the third vertex of the triangle?

Part B. Determine whether triangle $ABC$ is a right triangle based on its angle measures and side lengths.

Part C. If triangle $ABC$ is not a right triangle, can you classify what type of triangle $ABC$ is?

## Instructional Items

*Instructional Item 1*

Given $J$ $(-4,2)$ and $(2,1)$, find the coordinates of point $M$ on $\overline{JK}$ that partitions the segment into the ratio $1:2$.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

# MA.912.GR.3.4

## Benchmark

**MA.912.GR.3.4** Use coordinate geometry to solve mathematical and real-world problems involving perimeter or area of polygons.

*Example:* A new community garden has four corners. Starting at the first corner and working counterclockwise, the second corner is 200 feet east, the third corner is 150 feet north of the second corner and the fourth corner is 100 feet west of the third corner. Represent the garden in the coordinate plane, and determine how much fence is needed for the perimeter of the garden and determine the total area of the garden.

## Connecting Benchmarks/Horizontal Alignment

## Terms from the K-12 Glossary

- MA.912.GR.4.4
- MA.912.T.1.4

- Area
- Perimeter
- Polygon

## Vertical Alignment

**Previous Benchmarks**

- MA.2.GR.2
- MA.3.GR.2
- MA.4.GR.2
- MA.5.GR.2
- MA.6.GR.2
- MA.7.GR.1.1, MA.7.GR.1.2
- MA.8.GR.1.2
- MA.912.AR.2.1

**Next Benchmarks**

- MA.912.C.5.5

## Purpose and Instructional Strategies

In elementary grades, students are introduced to the concepts of perimeter and area, with a focus on rectangles. In middle grades, students expand their knowledge of areas of quadrilaterals and triangles. In Geometry, students find perimeter and area of two-dimensional polygons on the coordinate plane. In later courses, students develop further tools for finding area, especially using calculus.

- Instruction includes discussing the convenience of answering with exact values (e.g., the simplest radical form) or with approximations (e.g., rounding to the nearest tenth or hundredth). It is also important to explore the consequences of rounding partial answers on the accuracy or precision of the final answer, especially when working in real-world contexts.
- Problem types include polygons that are convex (where all interior angle measures are less than 180 degrees), concave (where at least one interior angle measure is more than 180 degrees), regular polygons (where all interior angle measures and side lengths are equivalent) and irregular polygons.
- Instruction includes the use of the distance formula and the Pythagorean Theorem to find side lengths of the polygon in order to determine the perimeter or area.
- Instruction includes various methods to determine area on the coordinate plane.
    - Decomposition
      Students can decompose the polygon into triangles and rectangles (or other quadrilaterals) to determine the total area by adding each of the partial areas.
    - Subtraction Method (Box Method)
      Students can draw a rectangle is that includes as many vertices of the polygon as possible and then to determine the area of the polygon, subtract the area(s) of the shape(s) that is (are) in the "box" but not in the polygon from the area of the rectangle.
    - For possible enrichment, other methods include, but are not limited to, Pick's Theorem or the Shoelace method.
- Refer to Appendix E for formulas of two-dimensional figures. It may be helpful to review the derivation of these formulas, as this was done in middle grades.

## Common Misconceptions or Errors

- Students may confuse the concepts of area and perimeter.

- Students may assume that the larger the area a polygon has, the larger its perimeter will be.
    - For example, a rectangle may have an area of 64 square meters and could have a perimeter of 32 meters or any perimeter larger than 32 meters.

## Instructional Tasks

*Instructional Task 1 (MTR.3.1, MTR.6.1)*

Given parallelogram $EFGH$ with vertices $E(-1,5)$, $F(2,8)$, $G(4,4)$ and $H(1,1)$.

Part A. Find the exact perimeter and area of the parallelogram.

Part B. Find the perimeter and area of the parallelogram to the nearest tenth.

*Instructional Task 2 (MTR.2.1, MTR.4.1)*

Joe's commute to work can be represented in the coordinate plane as follows:

- His house is at $H(0,0)$.
- His favorite coffee shop is at $C(7,6)$ where he stops every morning.
- His office is at $W(4,13)$.
- He goes back home from his office every day without stopping.

Part A. Assume that Joe lives in a city where the roads are parallel to the coordinate axes and each intersection occurs at integer coordinates. Represent his route on the coordinate plane where each city block is one coordinate unit by one coordinate unit, which measures 175 yards by 175 yards.

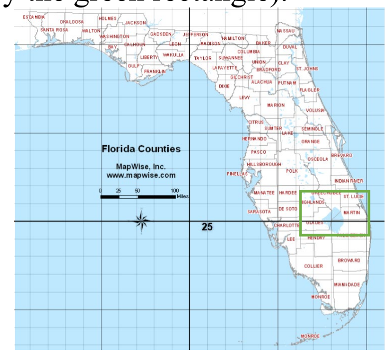
Part B. What is the total distance, in yards, that Joe commutes every day, assuming that he stays on the roads?

Part C. If Joe could take the most direct route (cutting across city blocks) for his commute, what would be his total distance, in yards, that he commutes every day?

## Instructional Items

*Instructional Item 1*

The Move With Us Run Team is planning a run around the combined perimeter of Polk and Osceola counties (as shown by the green rectangle).




Part A. What are the coordinates of the four vertices that could be used to measure the run around the two counties? Use the scale provided on the map to determine the coordinates.

Part B. Using the coordinates found in Part A, what would be the total distance of the run, in miles?

Part C. Assume that the group runs a total of 10 miles every day, how many days would it take them to complete the distance around the two counties?

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

**MA.912.GR.4** *Use geometric measurement and dimensions to solve problems.*

*MA.912.GR.4.1*

## Benchmark

| | |
|---|---|
| **MA.912.GR.4.1** | Identify the shapes of two-dimensional cross-sections of three-dimensional figures. |

Benchmark Clarifications:

*Clarification 1:* Instruction includes the use of manipulatives and models to visualize cross-sections.

*Clarification 2:* Instruction focuses on cross-sections of right cylinders, right prisms, right pyramids and right cones that are parallel or perpendicular to the base.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.1.5

## Terms from the K-12 Glossary

- Circle
- Cone
- Cylinder
- Prism
- Pyramid
- Rectangle
- Square
- Triangle

## Vertical Alignment

**Previous Benchmarks** **Next Benchmarks**

- MA.5.GR.1
- MA.6.GR.2.4
- MA.7.GR.2.1
- MA.912.GR.7
- MA.912.C.5.7

## Purpose and Instructional Strategies

In the elementary grades, students classified two- and three-dimensional figures. In middle grades, students worked with nets of three-dimensional figures when determining surface area. In Geometry, students begin to understand the concept of the two-dimensional cross-sections of familiar three-dimensional figures in preparation to discuss the formulas for the volume of these figures. In later courses, students will relate cross-sections to conic sections and will use cross-sections as a basis for finding the volumes of other three-dimensional figures.

- For the purposes of this benchmark, there is no expectation for students to master cross-sections of non-right, or oblique, three-dimensional figures. Instruction can include these as an enrichment or as comparison to cross-sections of right figures. As an additional enrichment, composite figures can also be utilized within instruction.
- Instruction includes the student understanding that among the shapes of the two-dimensional cross-sections can be circles, rectangles, squares, triangles, trapezoids and all other polygons that can be used as the base of right prisms and right pyramids (when the plane is parallel or perpendicular to the bases).
- When the vertical cross-section of a right cone does not include the apex, it is one piece of a hyperbola. Since many students at this level are not familiar with hyperbolas, the expectation is not to name the hyperbola, but be able to draw or visualize this cross-section. For enrichment, instruction may include showing that it is not a parabola, since it has diagonal asymptotes.
- Instruction focuses on pyramids with bases that are either equilateral triangles or squares, with the vertical cross-sections being parallel to a side of the base so that the vertical cross-sections are isosceles triangles and isosceles trapezoids. Since vertical cross-sections of pyramids include triangles that are not isosceles, and a variety of irregular polygons it may be difficult for students to visualize or name.
- Instruction includes utilizing objects, such as soda cans, cereal boxes or party hats, as models to explore their cross-sections. Additionally, students can explore other cross-sections using manipulatives such as clay and string to cut through the three-dimensional figure. (*MTR.7.1*)

## Common Misconceptions or Errors

- Students may oversimplify when they try to visualize cross-sections. To help address this misconception, use real-world three-dimensional figures to explore their cross-sections, as well as animations.
- Students may have difficulty with vertical cross-sections of pyramids and cones. To help address this, utilize manipulatives and physical models within instruction.

## Instructional Tasks

*Instructional Task 1 (MTR.3.1, MTR.4.1)*

Part A. Draw and name right three-dimensional figures that could have a triangular cross-section.

Part B. Draw and name right three-dimensional figures that could have a circular cross-section.

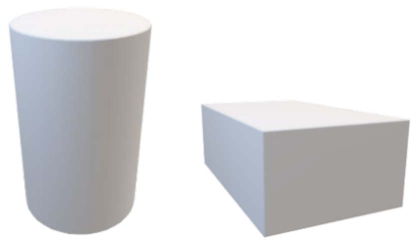
Part C. Compare your answers from Parts A and B with a partner.

*Instructional Task 2 (MTR.3.1)*

Part A. Fill in the blank below.

Both a right cylinder and a right prism have _______ cross-sections when cut perpendicular to the base.

Part B. Draw some cross-sections that are perpendicular to the base for each figure below.



## Instructional Items

*Instructional Item 1*

Which of the following polygons are cross-sections that are parallel or perpendicular to the base of a regular pentagonal pyramid? Select all that apply.

a. Triangle
b. Parallelogram
c. Trapezoid
d. Pentagon
e. Hexagon
f. Octagon

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

*MA.912.GR.4.2*

## Benchmark

**MA.912.GR.4.2** Identify three-dimensional objects generated by rotations of two-dimensional figures.

Benchmark Clarifications:

*Clarification 1:* The axis of rotation must be within the same plane but outside of the given two-dimensional figure.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.2.4, MA.912.GR.2.5

## Terms from the K-12 Glossary

- Circle
- Cone
- Cylinder
- Prism
- Pyramid
- Rectangle
- Square
- Triangle

## Vertical Alignment

**Previous Benchmarks**

- MA.5.GR.1.2

**Next Benchmarks**

- MA.912.C.5.7

## Purpose and Instructional Strategies

In grade 5, students identified three-dimensional figures. In Geometry, students visualize and discuss a three-dimensional figure as the result of a two-dimensional figure being rotated about an axis. In later courses, students will use rotations to find the volume of a three-dimensional figure (solid of revolution) using integrals. (*MTR.5.1*)

- Instruction begins with using a boundary line of a two-dimensional figure, including irregular figures, as the axis of the rotation, then move to an axis outside the two-dimensional figure.
  - For example, when rotating a rectangle about one of its boundary lines, a cylinder will be generated. If the rectangle is rotated about a line that is parallel to a side and not touching the figure, a cylinder with a hole down its center (a washer) will be generated.
  - For example, when rotating a right triangle about an axis containing one of the legs, a cone will be generated. When rotating a right triangle about the hypotenuse, a double cone will be generated.
- For enrichment purposes, include cases where an axis of symmetry is the axis of rotation; this will help make the connection of reflections to rotations about an axis.
  - For example, if a non-square rectangle is rotated about a line of symmetry, then the three-dimensional figure that would be generated is a cylinder.
- Instruction includes exploring irregular shapes that can be rotated to generate vases or exploring the rotation of a sphere about an axis that is outside to generate a torus.
- Problem types include two-dimensional figures presented on the coordinate plane with vertical and horizontal axis of rotation. When presented on the coordinate plane, students can identify some attributes of the three-dimensional object generated by the rotation like the height or the radius.
- Instruction includes the use of models, such as straws and cardboard, or animations.

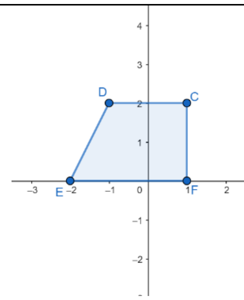
## Common Misconceptions or Errors

- Students may oversimplify when they try to visualize these rotations at first. To help address this, have students utilize physical models.

## Instructional Tasks

*Instructional Task 1 (MTR.4.1, MTR.5.1)*

Trapezoid $DCFE$ is shown on the coordinate plane below.




Part A. If the trapezoid is extended to create right triangle $EFB$, what are the coordinates of point $B$?
Part B. If triangle $EFB$ is rotated about line $x = 1$, what figure will it generate?
Part C. Determine the volume of the generated from Part B.
Part D. If trapezoid $DCFE$ is rotated about line $x = 1$, describe the figure that is generated.
Part E. Determine the volume of the generated from Part D.
Part F. If trapezoid $DCFE$ is rotated about line $x = 3$, what figure will it generate?
Part G. Determine the volume of the generated from Part F.

*Instructional Task 2 (MTR.3.1)*

Describe the figure that would be generated from the result of rotating a circle about a line outside the circle.

## Instructional Items

*Instructional Item 1*

Which real-world object could be used describe the figure generated by rotating a rectangle about a line that is parallel to a side but not touching the rectangle?

a. A doughnut
b. A piece of plastic tubing
c. An ice cream cone
d. A shoebox
e. An egg

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

*MA.912.GR.4.3*

## Benchmark

| | |
|---|---|
| **MA.912.GR.4.3** | Extend previous understanding of scale drawings and scale factors to determine how dilations affect the area of two-dimensional figures and the surface area or volume of three-dimensional figures. |

*Example:* Mike is having a graduation party and wants to make sure he has enough pizza. Which option would provide more pizza for his guests: one 12-inch pizza or three 6-inch pizzas?

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.1.6
- MA.912.GR.2.4, MA.912.GR.2.8

## Terms from the K-12 Glossary

- Area
- Scale Factor
- Scale Model

## Vertical Alignment

**Previous Benchmarks**

- MA.7.GR.1.5
- MA.8.GR.2.2, MA.8.GR.2.4
- MA.912.AR.2.1

**Next Benchmarks**

## Purpose and Instructional Strategies

In middle grades, students learned about scale drawings and scale factors. In Geometry, students use that previous knowledge to learn about how changes in the dimensions of a figure due to a dilation will affect the area of two-dimensional figures and the surface area or volume of three-dimensional figures in a way they can predict. *(MTR.2.1)* This understanding will be valuable to students in science courses.

- Instruction includes exploring the effect of changing the dimensions of two-dimensional and three-dimensional figures using different factors. It may be helpful to begin exploring through specific problems working with a table of values or with algebraic formulas.
  - For example, have students explore what happens to the area of a rectangle if the height is doubled and the length is tripled. Additionally, have them explore what happens to the volume of a cylinder if the height is multiplied by 0.5 and the radius is multiplied by 4.
- Instruction includes reviewing that the area of the image of a dilation with scale factor $k$ is $k^2$ times the area of the pre-image for any two-dimensional figure (as this was done in grade 7).
- Instruction includes the student understanding that the surface area of the image of a dilation with scale factor $k$ is $k^2$ times the surface area of the pre-image, and the volume of the image of a dilation with scale factor $k$ is $k^3$ times the volume of the pre-image for any three-dimensional figure.

### Common Misconceptions or Errors

- Students may multiply the area, surface area or volume by the scale factor instead of thinking about the multiple dimensions.
- Students may believe the scale factor has the same effect on surface area and volume. To help address this, discuss the effects on surface area using two-dimensional nets of simple figures and then compare to the effects on volumes.

### Instructional Tasks

*Instructional Task 1 (MTR.4.1, MTR.5.1)*

Use the table below to answer the following questions.

<table>
<tr><th>Original Square Pyramid</th><th>Dilation with scale factor $k$</th><th>New Surface Area</th><th>New Volume</th></tr>
<tr><td rowspan="3">Length of the base is 4 inches<br>Width of the base is 4 inches<br>Height of the pyramid is 2 inches<br><br>Surface Area = _______ sq. inches<br>Volume = _______ cubic inches</td><td>$k = 2$</td><td></td><td></td></tr>
<tr><td>$k = 3$</td><td></td><td></td></tr>
<tr><td>$k = \frac{1}{2}$</td><td></td><td></td></tr>
</table>

Part A. Determine the surface area and volume of the square pyramid.
Part B. Given the three different dilations, or scale factors, determine the new surface areas and volumes.
Part C. Compare each of the new surface areas to the original surface area. Compare each of the new volumes to the original volume.
Part D. Predict the surface area and volume of the square pyramid resulting from a dilation with a scale factor of 5? Explain the method you choose.

### Instructional Items

*Instructional Item 1*

The perfume Eau de Matimatica is packaged in a triangular prism bottle. The dimensions of the travel size are $\frac{1}{3}$ the dimensions of the standard bottle. How does the volume of the standard bottle compare to the travel size?

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

## *MA.912.GR.4.4*

### Benchmark

**MA.912.GR.4.4** Solve mathematical and real-world problems involving the area of two-dimensional figures.

*Example:* A town has 23 city blocks, each of which has dimensions of 1 quarter mile by 1 quarter mile, and there are 4,500 people in the town. What is the population density of the town?

Benchmark Clarifications:

*Clarification 1:* Instruction includes concepts of population density based on area.

### Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.1.6

### Terms from the K-12 Glossary

- Area

- MA.912.GR.3.4
- MA.912.T.1.2

## Vertical Alignment

**Previous Benchmarks**

- MA.5.GR.2
- MA.6.GR.2
- MA.7.GR.1.1, MA.7.GR.1.2
- MA.8.GR.1.2
- MA.912.AR.2.1

**Next Benchmarks**

- MA.912.C.5

## Purpose and Instructional Strategies

In elementary grades, students are introduced to the concepts of perimeter and area, with a focus on rectangles. In middle grades, students expand their knowledge of areas of quadrilaterals and triangles. In Geometry, students solve mathematical and real-world problems involving areas of two-dimensional figures, including population density. In Calculus, students will use integrals to connect the concept of area to many other real-world and mathematical contexts.

- Instruction includes reviewing units and conversions within and across different measurement systems (as this was done in middle grades).
- Instruction includes discussing the convenience of answering with exact values (e.g., the simplest radical form or in terms of pi) or with approximations (e.g., rounding to the nearest tenth or hundredth or using 3.14, $\frac{22}{7}$ or other approximations for pi). It is also important to explore the consequences of rounding partial answers on the accuracy or precision of the final answer, especially when working in real-world contexts.
- Instruction includes exploring the area of regular polygons and the formula based on the perimeter and the apothem ($A = \frac{1}{2}ap$, where $a$ is the length of the apothem and $p$ is the perimeter). The apothem is the line segment from the center to the midpoint of on the sides of a regular polygon. In many cases, finding the length of the apothem will require the use of trigonometric ratios.
- The population density based on area is calculated by the quotient of the total population and the total area. Have students practice finding the population density or the total population, given the dimensions of a two-dimensional figure. That is, part of their work includes finding the area based on the dimensions. *(MTR.7.1)*
- Instruction includes exploring a variety of real-world situations where finding the area is relevant for different purposes. Problem types include components like percentages, cost and budget, constraints, comparisons and others.
- Problem types include finding missing dimensions given the area of a two-dimensional figure or finding the area of composite figures.

### Common Misconceptions or Errors

- Students may not be careful with units of measurement involving area, particularly when converting from one unit to another.
  - For example, since there are 100 centimeters in a meter, a student may incorrectly conclude that there are 100 square centimeters in a square meter.

### Instructional Tasks

*Instructional Task 1 (MTR.7.1)*

In 2019, the population of Leon County was 293,582 and the population of Sarasota County was 433,742. The area of Sarasota County is 752 square miles, while the area of Leon County is 702 square miles.

Part A. Which county has a higher population density?

Part B. If the physical shape of the county identified in Part A was a rectangle, what are possible dimensions of the county if the length is greater than the width?

Part C. If the county identified in Part A was the physical shape of a right triangle, what are possible dimensions of the base and height of the county?

Part D. Does changing the shape of the tract of land change the population density of the county?

*Instructional Task 2 (MTR.3.1)*

The area of a regular decagon is 24.3 square meters. Determine the side length, in meters, of the regular decagon.

### Instructional Items

*Instructional Item 1*

In 2019, the population for Siesta Key, FL, was 5,573 while Destin, FL, had a population of 13,702. Siesta Key is 3.475 square miles and Destin is 8.46 square miles. Which location has a smaller population density?

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

## *MA.912.GR.4.5*

### Benchmark

| | |
|---|---|
| **MA.912.GR.4.5** | Solve mathematical and real-world problems involving the volume of three-dimensional figures limited to cylinders, pyramids, prisms, cones and spheres. |

*Example:* A cylindrical swimming pool is filled with water and has a diameter of 10 feet and height of 4 feet. If water weighs 62.4 pounds per cubic foot, what is the total weight of the water in a full tank to the nearest pound?

Benchmark Clarifications:

*Clarification 1:* Instruction includes concepts of density based on volume.

*Clarification 2:* Instruction includes using Cavalieri's Principle to give informal arguments about the formulas for the volumes of right and non-right cylinders, pyramids, prisms and cones.

### Connecting Benchmarks/Horizontal Alignment

- MA.912.T.1.2

### Terms from the K-12 Glossary

- Cone
- Cylinder
- Prism

- Pyramid
- Sphere

## Vertical Alignment

**Previous Benchmarks**

- MA.6.GR.2.3
- MA.7.GR.2.3
- MA.912.AR.2.1

**Next Benchmarks**

- MA.912.C.5.7

## Purpose and Instructional Strategies

In middle grades, students determined the volume of right rectangular prisms and right cylinders. In Geometry, students explore for the first time the volume of pyramids, cones, and spheres. In later courses, student learn more advanced methods for calculating volume.

- Instruction includes reviewing units and conversions within and across different measurement systems (as this was done in middle grades).
- Instruction includes discussing the convenience of answering with exact values (e.g., the simplest radical form or in terms of pi) or with approximations (e.g., rounding to the nearest tenth or hundredth or using 3.14, $\frac{22}{7}$ or other approximations for pi). It is also important to explore the consequences of rounding partial answers on the accuracy or precision of the final answer, especially when working in real-world contexts.
- Instruction includes reviewing the definition of cylinders, pyramids, prisms, cones and spheres (as this was done in grade 5), and discussing the definitions of right and oblique polyhedrons, cubes, tetrahedrons, regular prisms and regular pyramids.
- The population or material density based on volume is calculated by the quotient of the total population or material and the volume (i.e., population density of fish in a spherical aquarium or density of salt in a bucket of water). Have students practice finding the population or material density or the total population or material amount, given the dimensions of a three-dimensional figure. That is, part of their work includes finding the volume based on the dimensions. *(MTR.7.1)*
- Instruction includes the connection to two-dimensional cross-sections of three-dimensional figures to explore Cavalieri's Principle, which states that if in two solids of equal height, the cross-sections made by planes parallel to and at the same distance from their respective bases are always equal, then the volumes of the two solids are equal. *(MTR.5.1)*
    - For example, have students compare the volume of two stacks of pennies of the same height, one organized in a straight column and the other one, one penny on top of the other, but in a slanted stack. Discuss the shape of their cross-sections at the same height and what happens with their volumes.
    - For example, have students discuss how this principle is applied in the calculation of volumes of non-right (oblique) three-dimensional figures.
    - For example, have students discuss how this principle can be used to find the volume of a non-right cylinder given a right cylinder with the same height and same cross-sections. (*MTR.4.1*)
- Instruction includes exploring a variety of real-world situations where finding the volume or volume density is relevant for different purposes. Problem types include components like percentages, cost and budget, constraints, comparisons, BTUs, nutrition (e.g.,

calories per cup), moisture content (e.g., ounces of water in a gallon of honey) or others.

- Problem types include finding missing dimensions given the volume of a three-dimensional figure or finding the volume of composite figures.

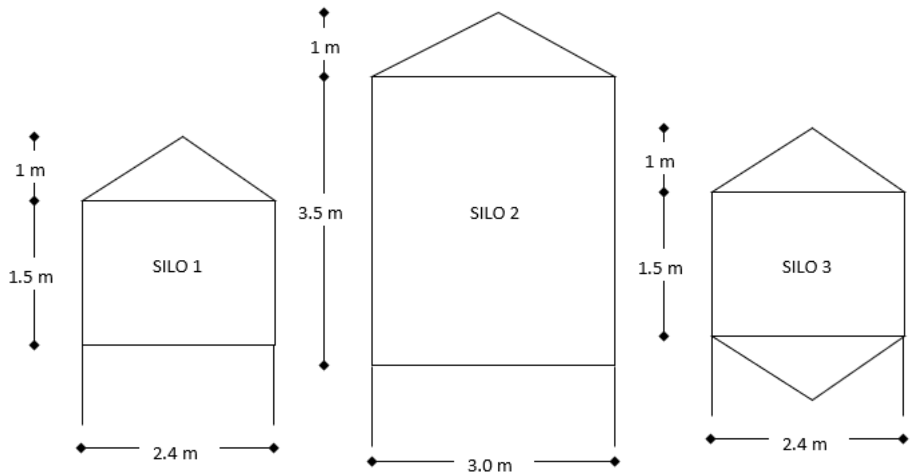
## Common Misconceptions or Errors

- Students may not be careful with units of measurement involving volume, particularly when converting from one unit to another.
    - For example, since there are approximately 25.4 millimeters in an inch, a student may incorrectly conclude that there are 25.4 cubic millimeters in a cubic inch.

## Instructional Tasks

*Instructional Task 1 (MTR.7.1)*

When filling cylindrical silos, the top cone is not filled. However, if the silo has a bottom cone, it is filled. Three different silos are shown in the image below.




Part A. In silo 3, the top and bottom cones are congruent. How much more grain could silo 3 hold than silo 1?

Part B. The diameter of silo 1 is 80% the diameter of silo 2. Is the capacity of silo 1 80% the capacity of silo 2?

*Instructional Task 2 (MTR.4.1)*

The radius of a sphere is 4 units so its volume is $\frac{256}{3}\pi$ cubic units.

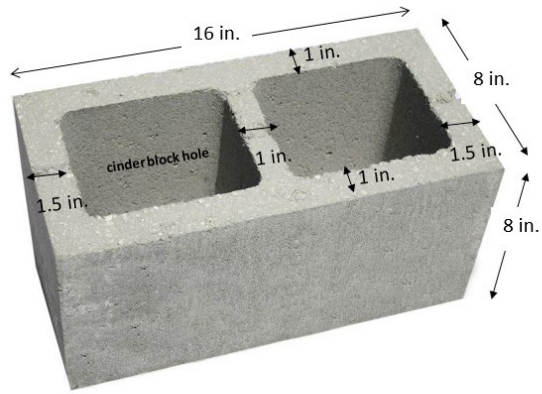
Part A. Discuss the value of this kind of answer for its accuracy and precision.

Part B. Discuss the effect of replacing $\pi$ in the formulas with 3.14, 3.1416, $\frac{22}{7}$ and other approximations. What happens with the answer, the volume of the figure, in each case?

## Instructional Items

*Instructional Item 1*

Joshua is going to create a garden border around three sides of his backyard deck using cinder blocks. He is going to plant a flower in each hole of the cinder block. The dimensions of the cinder blocks are 8 inches by 16 inches by 8 inches. Each hole needs to be completely filled with potting soil before the flowers can be planted. Potting soil is sold in 1 cubic foot bags.




Part A. What are the dimensions of a cinder block hole?
Part B. The patio is a square with a side length of 8 feet. One of the sides of the square patio is adjacent to an exterior wall of the house. If Joshua puts blocks around the other three sides of the patio, how many bags will Joshua need to purchase to fill the blocks?

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

## *MA.912.GR.4.6*

### Benchmark

**MA.912.GR.4.6** Solve mathematical and real-world problems involving the surface area of three-dimensional figures limited to cylinders, pyramids, prisms, cones and spheres.

### Connecting Benchmarks/Horizontal Alignment

- MA.912.T.1.2

### Terms from the K-12 Glossary

- Circle
- Cone
- Cylinder
- Prism
- Pyramid
- Rectangle
- Sphere
- Square
- Triangle

### Vertical Alignment

**Previous Benchmarks**

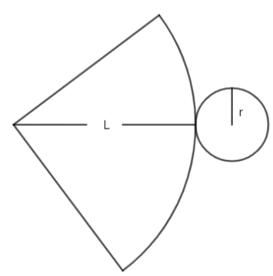
- MA.6.GR.2.4
- MA.7.GR.2.1, MA.7.GR.2.2
- MA.912.AR.2.1

**Next Benchmarks**

### Purpose and Instructional Strategies

In middle grades, students determined surface area using nets and formulas for right rectangular prisms, right rectangular pyramids and right circular cylinders. In Geometry, students explore for the first time the surface area of cones and spheres.

- Instruction includes reviewing units and conversions within and across different measurement systems (as this was done in middle grades).
- Instruction includes discussing the convenience of answering with exact values (e.g., the simplest radical form or in terms of pi) or with approximations (e.g., rounding to the nearest tenth or hundredth or using 3.14, $\frac{22}{7}$ or other approximations for pi). It is also important to explore the consequences of rounding partial answers on the accuracy or precision of the final answer, especially when working in real-world contexts.
- Instruction includes reviewing the definition of cylinders, pyramids, prisms, cones and spheres (as this was done in grade 5), and discussing the definitions of right and oblique polyhedrons, cubes, tetrahedrons, regular prisms and regular pyramids.
- Instruction includes the connection to finding areas of two-dimensional figures to determine the surface area of cylinders, pyramids, prisms and cones.
    - For example, the surface area of a cylinder is the result of combining the area of the bases (circles with radius $r$) with the lateral area (a rectangle with base $C = 2\pi r$ and height equal to the height of the cylinder). The area of the circles is $\pi r^2$ and the area of the rectangle is $bh$ which is equivalent to $Ch$ which is equivalent to $2\pi rh$. Therefore, given a cylinder with radius $r$ and height $h$, its surface area is $2\pi r^2 + 2\pi rh$.
    - For example, the surface area of a cone is the result of combining the area of the base (a circle with radius $r$ and circumference $C$) and the area of the curved surface (a circular sector with radius $L$, which is the slant height of the cone and arc length $C$). The slant height $L$ is $\sqrt{h^2 + r^2}$, where $h$ is the height of the cone. The area of the circle is $\pi r^2$ and the area of the circular sector is $\frac{1}{2}(arc\ length)(radius)$, which is equivalent to $\frac{1}{2}CL$, which is equivalent to $\frac{1}{2}(2\pi r)L$, which is equivalent to $\pi rL$. Therefore, given a cone with radius $r$ and height $h$, its surface area if $\pi r^2 + \pi rL = \pi r^2 + \pi r\sqrt{h^2 + r^2}$ is $\pi r(r + \sqrt{h^2 + r^2})$.




- Instruction includes exploring the surface area of cylinders, pyramids and prisms as the result of combining areas of triangles, rectangles and circles (and when needed, other polygons). Students should understand the similarities and differences between lateral area and surface area. (*MTR.2.1*)
- Since deriving the surface area of a sphere requires Calculus, students will not be able to explore its formula and can be calculated using the formula $SA = 4\pi r^2$.
- Instruction includes exploring a variety of real-world situations where finding the surface area is relevant for different purposes. Problem types include components like percentages, cost and budget, constraints, comparisons, or others.

- Problem types include finding missing dimensions given the surface area of a three-dimensional figure, finding the surface area of composite figures or determining which face to include in calculations within real-world context (i.e., the surface area required to paint a house, the surface area that will be covered by a label in a soup can).

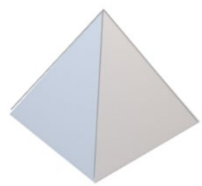
### Common Misconceptions or Errors

- Students may have trouble working with formulas by making incorrect substitutions or incorrect use of the order of operations.

### Instructional Tasks

*Instructional Task 1 (MTR.7.1)*

There are three Pyramids of Giza. The largest, the Great Pyramid, has an approximately square base with side lengths averaging 230 meters and a lateral surface area of 85,836 square meters. What is the height of the Great Pyramid?



*Instructional Task 2 (MTR.4.1)*

The surface area of a sphere with radius 10 is $400\pi$ square units.

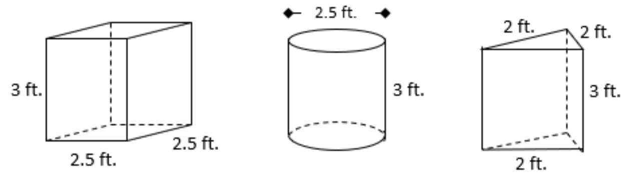
Part A. Discuss the value of this kind of answer for its accuracy and precision.

Part B. Discuss the effect of replacing $\pi$ in the formulas with $3.14$, $3.1416$, $\frac{22}{7}$ and other approximations. What happens with the answer, the surface area of the figure, in each case?

## Instructional Items

*Instructional Item 1*

Kristin and Rachel are hosting an art show where they will showcase local artists' sculptures. They are painting pedestals upon which the sculptures will be placed. Pictures of the pedestals they will be using are below. One gallon of paint can cover 400 square feet.




Part A. How many gallons of paint will they need to purchase to cover at least 4 of each type of pedestal? Assume that the base of each will not be painted.

Part B. If there is any paint left over, determine how many of which shape pedestals could be painted.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

**MA.912.GR.5** *Make formal geometric constructions with a variety of tools and methods.*

*MA.912.GR.5.1*

## Benchmark

**MA.912.GR.5.1** Construct a copy of a segment or an angle.

Benchmark Clarifications:
*Clarification 1:* Instruction includes using compass and straightedge, string, reflective devices, paper folding or dynamic geometric software.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.1.1, MA.912.GR.1.2
- MA.912.GR.2.2, MA.912.GR.2.3, MA.912.GR.2.5
- MA.912.LT.4.8

## Terms from the K-12 Glossary

- Angle

## Vertical Alignment

**Previous Benchmarks**

- MA.2.GR.1.1
- MA.3.GR.1.1
- MA.4.GR.1.2

**Next Benchmarks**

## Purpose and Instructional Strategies

In elementary grades, students drew lines and angles using a variety of tools, including rulers and protractors. In Geometry, students are introduced to constructions for the first time, specifically copying a segment or an angle. These two procedures are embedded in other basic constructions, and the concept of constructing and identifying copies of segments and angles is closely connected to visualizing and understanding congruence.

- Instruction includes the use of manipulatives, tools and geometric software. Allowing students to explore constructions with dynamic software reinforces why the constructions work.
  - For example, students can use tracing/folding paper (e.g., patty paper) to trace the copy of an angle, or the copy of a segment, and verify that the angle and its copy are congruent, or that the segment and its copy are congruent. Additionally, using several folds, it is possible to verify the congruency of two angles or two segments drawn on the same piece of paper.
- Instruction includes the connection to logical reasoning and visual proofs when verifying that a construction works.
- Instruction includes discussing the role of the compass in a geometric construction, beyond drawing circles, and how a string can replace a compass. Most of the time in this course, compasses will be used to draw arcs. Discuss how no matter the point chosen on the arc, the distance to the given point is the same.
  - For example, students can place the compass at $P$ and draw an arc. Choosing two points on the arc, $A$ and $B$, the distance to $P$ is the same, $AP = BP$ and $\overline{AP} \cong \overline{BP}$. $\overline{AP}$ and $\overline{BP}$ are radii of the circle containing the drawn arc centered at $P$.
- Instruction includes the student understanding that in a geometric construction, one does not use the markings on a ruler or on a protractor to copy a segment or angle. Students should realize that there are limitations on precision that are inherent in the markings on rulers or protractors.
- It is important to build the understanding that formal constructions are valid when the lengths of segments or measures of angles are not known, or have values that do not appear on a ruler or protractor, including irrational values.
- For expectations of this benchmark, constructions should be reasonably accurate and the emphasis is to make connections between the construction steps and the definitions, properties and theorems supporting them.
- While going over the steps of geometric constructions, ensure that students develop vocabulary to describe the steps precisely. *(MTR.4.1)*
- Problem types include identifying the next step of a construction, a missing step in a construction or the order of the steps in a construction.

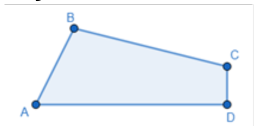
## Common Misconceptions or Errors

- Students may not understand that the size of the angle, "the opening," is what is being measured when copying an angle.
- Students may not understand why they are not using the marking on rulers and protractors to copy segments and angles.

## Instructional Tasks

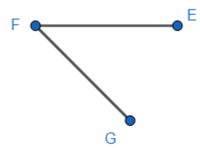
*Instructional Task 1 (MTR.2.1, MTR.3.1)*

Create a construction of quadrilateral $JKLM$ so that it is congruent to quadrilateral $ABCD$.




*Instructional Task 2 (MTR.2.1, MTR.5.1)*

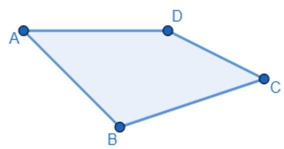
Given angle $EFG$ below, create a copy so that it creates parallelogram $EFGH$.

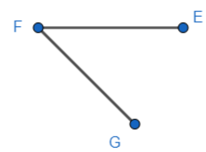



## Instructional Items

*Instructional Item 1*

Construct the necessary segments and angles to construct quadrilateral $EFGH$ so that it is congruent to quadrilateral $DABC$. Assume $\angle DAB \cong \angle EFG$, $\overline{DA} \cong \overline{EF}$ and $\overline{AB} \cong \overline{FG}$.




**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

*MA.912.GR.5.2*

## Benchmark

**MA.912.GR.5.2** Construct the bisector of a segment or an angle, including the perpendicular bisector of a line segment.

Benchmark Clarifications:

*Clarification 1:* Instruction includes using compass and straightedge, string, reflective devices, paper folding or dynamic geometric software.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.1.1, MA.912.GR.1.2
- MA.912.GR.2.2, MA.912.GR.2.3, MA.912.GR.2.5
- MA.912.LT.4.8

## Terms from the K-12 Glossary

- Angle

## Vertical Alignment

**Previous Benchmarks**

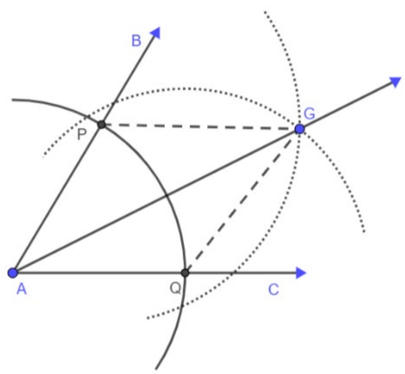
- MA.2.GR.1.1
- MA.3.GR.1.1
- MA.4.GR.1.2

**Next Benchmarks**

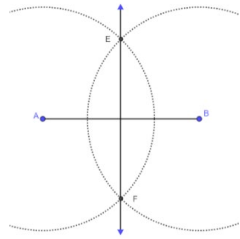
## Purpose and Instructional Strategies

In elementary grades, students drew lines and angles using a variety of tools, including rulers and protractors, and by making measurements with those tools, they could bisect lines and angles. In Geometry, students are introduced to constructions that do not rely on making measurements, specifically bisecting an angle and bisecting a segment, including perpendicular bisectors, using a compass and straightedge. These two procedures are embedded within constructing an inscribed circle and a circumscribed circle of a triangle as well as in the construction of a square inscribed in a circle.

- Instruction includes the use of manipulatives, tools and geometric software. Allowing students to explore constructions with dynamic software reinforces why the constructions work.
  - For example, the use of paper folding (e.g., patty paper) can be used to determine the angle bisector of a given angle and the midpoint or perpendicular bisector of a given segment.
- Instruction includes the connection to triangle congruence when constructing an angle bisector.
  - For example, have students place the compass at point $A$ and draw an arc intersecting the sides of the angle resulting in the points of intersection $P$ and $Q$. Students should realize that $\overline{AP} \cong \overline{AQ}$. Without changing the compass setting, add two arcs intersecting in the interior of the angle at the point $G$. Students should realize that $\overline{PG} \cong \overline{QG}$. By the Reflexive property of congruence, $\overline{AG} \cong \overline{AG}$. Therefore, $\Delta APG \cong \Delta AQG$ by SSS and since corresponding parts of congruent triangles are congruent (CPCTC), $\angle PAG \cong \angle QAG$ and $\overrightarrow{AG}$ is the angle bisector of $\angle BAC$.




- Instruction includes the connection to the converse of the Perpendicular Bisector Theorem when constructing a perpendicular bisector.
  - For example, students can set the compass width more than half the length of $\overline{AB}$. Students can draw arcs intersecting above and below the segment at points $E$ and $F$. Therefore, $\overline{AE} \cong \overline{BE}$ and $\overline{AF} \cong \overline{BF}$. That is, points $E$ and $F$ are each the same distance to the endpoints of $\overline{AB}$ and that means they lie on the Perpendicular Bisector.




- Instruction includes the student understanding that when one has constructed the perpendicular bisector, they have also constructed the midpoint of a segment. *(MTR.2.1)*
  - For example, using the same steps as in the last construction, the midpoint of the segment can be identified as the point where the perpendicular bisector meets the segment.
- Instruction includes the connection to logical reasoning and visual proofs when verifying that a construction works.
  - For example, once the construction of the perpendicular bisector is completed, discuss with students how this construction and a compass can be used to experimentally check the Perpendicular Bisector Theorem. (*MA.912.GR.1.1*)
- For expectations of this benchmark, constructions should be reasonably accurate and the emphasis is to make connections between the construction steps and the definitions, properties and theorems supporting them.
- While going over the steps of geometric constructions, ensure that students develop vocabulary to describe the steps precisely. *(MTR.4.1)*
- Instruction includes the connection between constructions and properties of quadrilaterals, including rhombi.
  - For example, when constructing the angle bisector, if the compass width is not changed throughout the process, then quadrilateral $AHGF$ is a rhombus since it has 4 equal sides ($\overline{AH}, \overline{HG}, \overline{GF}, \overline{FA}$). The diagonals of a rhombus bisect opposite angles. Therefore, $\angle HAF$ is bisected by the diagonal of the rhombus $\overline{AG}$ and $\overrightarrow{AG}$ is the angle bisector of $\angle BAC$. Similarly, when constructing the perpendicular bisector, it can be seen that the diagonals of a rhombus are perpendicular.
- Instruction includes the student understanding that in a geometric construction, one does not use the markings on a ruler or on a protractor to bisect a segment or angle. Students should realize that there are limitations on precision that are inherent in the markings on rulers or protractors.
- It is important to build the understanding that formal constructions are valid when the lengths of segments or measures of angles are not known, or have values that do not appear on a ruler or protractor, including irrational values.
- Problem types include identifying the next step of a construction, a missing step in a construction or the order of the steps in a construction.

## Common Misconceptions or Errors

- Students may not make the connection that any point on the perpendicular bisector is equidistant from the endpoints of the segment, not just the midpoint of the segment.
- Students may not understand why they are not using rulers and protractors to bisect segments and angles.

## Instructional Tasks

*Instructional Task 1 (MTR.7.1)*

A map of some popular universities is shown below.


Part A. Prove that Georgia Tech is approximately equidistant from Clemson University and Auburn University.

Part B. Find one or more universities that are approximately equidistant from Florida State University and Oklahoma State University?

## Instructional Items

*Instructional Item 1*

An image is provided below.


Part A. Construct the bisector of angle $D$.

Part B. Construct the midpoint of segment $DB$.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

## Benchmark

**MA.912.GR.5.3** Construct the inscribed and circumscribed circles of a triangle.

Benchmark Clarifications:
*Clarification 1:* Instruction includes using compass and straightedge, string, reflective devices, paper folding or dynamic geometric software.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.6.3
- MA.912.LT.4.8

## Terms from the K-12 Glossary

- Inscribed Circle
- Circumscribed Circle
- Triangle

## Vertical Alignment

**Previous Benchmarks**

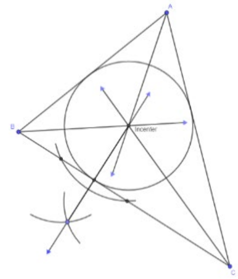
- MA.7.GR.1.4

**Next Benchmarks**

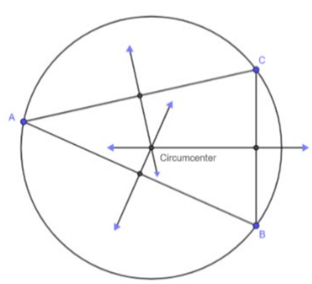
## Purpose and Instructional Strategies

In grade 7, students used a relationship between triangles and circles to understand the formula for the area of a circle. In Geometry, students identify and construct two special circles that are associated with a triangle.

- Instruction includes the use of manipulatives, tools and geometric software. Allowing students to explore constructions with dynamic software reinforces why the constructions work.
- Instruction includes the student understanding that in a geometric construction, one does not use the markings on a ruler or on a protractor to construct inscribed and circumscribed circles of a triangle. Students should realize that there are limitations on precision that are inherent in the markings on rulers or protractors.
- It is important to build the understanding that formal constructions are valid when the lengths of segments or measures of angles are not known, or have values that do not appear on a ruler or protractor, including irrational values.
- Instruction includes the connection to logical reasoning and visual proofs when verifying that a construction works.
- Instruction includes the connection to constructing angle bisectors and perpendicular bisectors. (*MTR.2.1*)
- Instruction includes using various methods, like the one described below, to construct an inscribed circle.
  - For example, given triangle $ABC$, students can construct two of the three angle bisectors to create their point of intersection, $D$. Students should realize that the point $D$ is the incenter of the triangle and may predict that point $D$ will be the center of the inscribed circle. To prove this prediction, students will need to prove that point $D$ is equidistant from each of the three sides. In order to prove this, students can construct the perpendicular segments from point $D$ to each of the three sides, and show that all three segments are congruent using triangle congruence criteria and D is the intersection of the angle bisectors. Each of these segments will be a radius of the inscribed circle, with the center of the circle at point $D$.



- When constructing an inscribed circle, students should make the connection to constructing perpendicular bisectors when they need to construct a line through the incenter of the triangle that is perpendicular to a side of the triangle.
  - For example, to construct such a line, students can place the compass at the incenter, point $D$, and draw arcs to determine two points, $E$ and $F$, on one of the sides. These points are equidistant to $D$. Then they using the same compass setting, place the compass at $E$ and at $F$ and draw arcs intersecting on the opposite side of $\overline{EF}$ from $D$. The intersection of these arcs, $Q$, is the same distance to $E$ and to $F$. Therefore, the line passing thru $D$ and $Q$ is the perpendicular bisector of $\overline{EF}$, so it is also a line perpendicular to the side of the triangle.
- Students should understand that the shortest segment from a point, $D$, to a line is the segment from $D$ to the line that is perpendicular to the line. Additionally, students should understand that the circle centered at point $D$, which has this segment as a radius, is tangent to the line.
- Instruction includes using various methods, like the one described below, to construct a circumscribed circle.
  - For example, given triangle $ABC$, students can construct two of the three perpendicular bisectors of the sides of the triangle to create their point of intersection, $D$. Students should realize that the point $D$ is the circumcenter of the triangle and may predict that point $D$ will be the center of the circumscribed circle. To prove this prediction, students will need to prove that point $D$ is equidistant from each of the three vertices. In order to prove this, students can use the fact that point $D$ is the intersection of the perpendicular bisectors. Each of these segments will be a radius of the circumscribed circle, with the center of the circle at point $D$. So, to construct the circumscribed circle, one can set the compass equal to the distance between point $D$ and any one of the vertices and then draw the circle centered at point $D$.




- Instruction includes exploring the construction of circumscribed circles about various triangles. Have students explore acute, right and obtuse triangles, and compare the locations of the circumcenter of each. Students should understand that with a right triangle, the circumcenter is located at the midpoint of the hypotenuse.

- For expectations of this benchmark, constructions should be reasonably accurate and the emphasis is to make connections between the construction steps and the definitions, properties and theorems supporting them.
- While going over the steps of geometric constructions, ensure that students develop vocabulary to describe the steps precisely. *(MTR.4.1)*
- Problem types include identifying the next step of a construction, a missing step in a construction or the order of the steps in a construction.

**Common Misconceptions or Errors**

- Students may think that the when constructing a circumscribed circle, the center of the circle cannot be outside the triangle.

**Instructional Tasks**

*Instructional Task 1 (MTR.2.1, MTR.4.1)*

Part A. Construct angle bisectors for the three interior angles of a triangle using folding paper, a compass and straightedge and geometric software. What do you notice about each method of construction?

Part B. Repeat Part A with a triangle that is obtuse, isosceles, acute and right. Describe your findings.

Part C. Using the incenter as the center, a circle can be constructed inscribed in the triangle. How can you determine the radius of that circle, the inscribed circle?

Part D. Construct the inscribed circle of one of the triangles from Part B.

*Instructional Task 2 (MTR.2.1, MTR.4.1)*

Part A. Construct perpendicular bisector for the three sides of a triangle using folding paper, a compass and straightedge and geometric software. What do you notice about each method of construction?

Part B. Repeat Part A with a triangle that is obtuse, isosceles, acute and right. Describe your findings.

Part C. Using the circumcenter as the center, a circle can be constructed circumscribed about the triangle. How can you determine the radius of that circle, the circumscribed circle?

Part D. Construct the circumscribed circle of one of the triangles from Part B.

*Instructional Task 3 (MTR.5.1)*

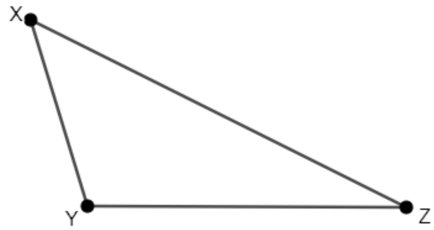
Part A. Given the line $l$ and the point $P$ external to the line $l$, construct a perpendicular line, $m$, through point $P$.

Part B. Use the construction from Part A to construct a line, $n$, that is parallel to the line $l$ and contains the point $P$.

## Instructional Items

*Instructional Item 1*

Construct the circle that is circumscribed about $\Delta XYZ$.




**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

*MA.912.GR.5.4*

## Benchmark

| | |
|---|---|
| **MA.912.GR.5.4** | Construct a regular polygon inscribed in a circle. Regular polygons are limited to triangles, quadrilaterals and hexagons. |

Benchmark Clarifications:
*Clarification 1:* When given a circle, the center must be provided.
*Clarification 2:* Instruction includes using compass and straightedge, string, reflective devices, paper folding or dynamic geometric software.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.1.4
- MA.912.GR.6.3
- MA.912.LT.4.8

## Terms from the K-12 Glossary

- Inscribed Polygon in a Circle

## Vertical Alignment

**Previous Benchmarks**

- MA.8.GR.1.6

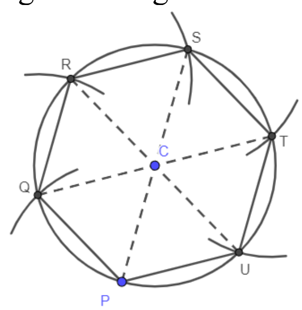
**Next Benchmarks**

## Purpose and Instructional Strategies

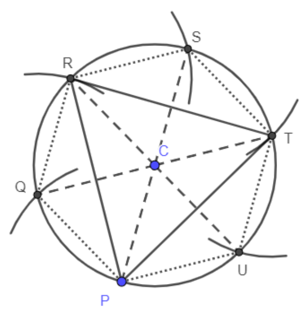
In grade 8, students learned about regular polygons and their interior angles. In Geometry, students inscribe such polygons in circles using compass and straightedge.

- Instruction includes the use of manipulatives, tools and geometric software. Allowing students to explore constructions with dynamic software reinforces why the constructions work.
- Instruction includes the student understanding that in a geometric construction, one does not use the markings on a ruler or on a protractor to construct inscribed and circumscribed circles of a triangle. Students should realize that there are limitations on precision that are inherent in the markings on rulers or protractors.
- It is important to build the understanding that formal constructions are valid when the lengths of segments or measures of angles are not known, or have values that do not appear on a ruler or protractor, including irrational values.
- Instruction includes the connection to logical reasoning and visual proofs when verifying that a construction works.

- To construct a square inscribed in a circle given the center of the circle, the procedure can start by drawing one of the diameters of the circle. This diameter will be one of the diagonals of the square. Then, students can construct the perpendicular bisector of the drawn diameter. The perpendicular bisector will intersect the circle in two points resulting in another diameter of the circle that is congruent and perpendicular to the drawn diameter. Students should realize that diagonals of squares have the same length and are perpendicular bisectors of one another. Therefore, the two diameters are the diagonals of the square to be constructed in the circle. To construct this square, draw the sides by connecting each of the endpoints of the diagonals.
- The construction of a regular hexagon inscribed in a circle (given the center) can start by choosing one point, $P$, on the circle, then, with the compass setting equal to the radius of the circle, students can draw an arc intersecting the circle at the point $Q$. Students should realize that the two points on the circle and the center of the circle are the vertices of an equilateral triangle. Students can then move the compass to the point $Q$ and draw another arc intersecting the circle at the point $R$. Students can repeat this process as they move around the circle until the arrive back at the point $P$. Students should realize that as they are repeating this process around the circle, they are forming six equilateral triangles. These triangles compose the regular hexagon that is inscribed in the circle.




- The construction of an equilateral triangle can begin by constructing a regular hexagon, as described above. Once the six equilateral triangles have been formed, to construct an equilateral triangle in the circle, students can join three pairs of alternating vertices (as shown below).




- Instruction includes the connection to the sum of the measures of the interior angles of a regular polygon to the construction of equilateral triangles and hexagons inscribed in a circle.
- Enrichment of this benchmark includes constructing other regular polygons, such as octagons and dodecagons, using angle bisection.
- For expectations of this benchmark, constructions should be reasonably accurate and the emphasis is to make connections between the construction steps and the definitions,

properties and theorems supporting them.

- While going over the steps of geometric constructions, ensure that students develop vocabulary to describe the steps precisely. *(MTR.4.1)*
- Problem types include identifying the next step of a construction, a missing step in a construction or the order of the steps in a construction.

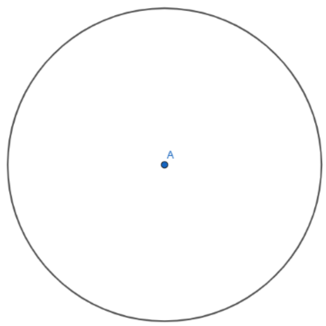
## Common Misconceptions or Errors

- Students may think or determine that when constructing a regular hexagon, they will not have all equal side lengths. To help address this, reiterate the importance of precision and accuracy when using a compass.

## Instructional Tasks

*Instructional Task 1*

Circle $A$ is provided below.




Part A. What do you know about all the points on circle $A$ in relation to point $A$?

Part B. Draw a point $J$ on circle $A$. Open the compass to the length of the resulting radius of circle $A$.

Part C. With the compass point on point $J$, arc on both sides of $J$ making sure to intersect circle $A$. Label the points of intersection as $K$ and $M$.

Part D. Classify triangle $AKJ$ by sides.

Part E. What is the measure of angle $KJM$?

Part F. What is the measure of an interior angle of a regular hexagon?

Part G. What process could be used to continue constructing a regular hexagon that is inscribed in circle $A$?

Part H. How could the construction of an inscribed regular hexagon be used to construct an inscribed equilateral triangle?

*Instructional Task 2 (MTR.3.1, MTR.4.1, MTR.5.1)*

Part A. Construct a regular hexagon inscribed in a circle.

Part B. Prove that the constructed hexagon is a regular hexagon.

Part C. Prove that if three pairs of alternating vertices of the hexagon from Part A are joined, it creates an equilateral triangle.

Part D. Compare your proof from Part C with a partner.

## Instructional Items

*Instructional Item 1*

Describe the steps to construct a square inscribed in circle $J$.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

*MA.912.GR.5.5*

## Benchmark

**MA.912.GR.5.5** Given a point outside a circle, construct a line tangent to the circle that passes through the given point.

Benchmark Clarifications:
*Clarification 1:* When given a circle, the center must be provided.
*Clarification 2:* Instruction includes using compass and straightedge, string, reflective devices, paper folding or dynamic geometric software.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.1.3
- MA.912.GR.3.3
- MA.912.GR.6.1

## Terms from the K-12 Glossary

- Circle

## Vertical Alignment

**Previous Benchmarks**

- MA.7.GR.1.3, MA.7.GR.1.4

**Next Benchmarks**

## Purpose and Instructional Strategies

In grade 7, students learned about the circumference and area of a circle. In Geometry, students learn about tangents to circles, including using a compass and straightedge to construct a tangent that passes through a given point outside of the circle.

- Instruction includes the use of manipulatives, tools and geometric software. Allowing students to explore constructions with dynamic software reinforces why the constructions work.
- Instruction includes the student understanding that in a geometric construction, one does not use the markings on a ruler or on a protractor to construct inscribed and circumscribed circles of a triangle. Students should realize that there are limitations on precision that are inherent in the markings on rulers or protractors.
- It is important to build the understanding that formal constructions are valid when the lengths of segments or measures of angles are not known, or have values that do not appear on a ruler or protractor, including irrational values.
- Instruction includes the connection to logical reasoning and visual proofs when verifying that a construction works.
- Instruction includes the connection to inscribed angles in circles; to the lengths of tangents drawn from an external point to a circle; to circumcenters of right triangles; and to a tangent of a circle (understanding that it is perpendicular to the radius at the point of tangency). (*MTR.5.1*)
- To construct line tangent to a circle that passes through the given point, students can start by drawing a segment from the center, $C$, to an external point, $P$. Students can then determine the midpoint, M, of $\overline{CP}$ by constructing the perpendicular bisector. Then, students can set the compass width equal to the distance from point $C$ to point $M$ and place the compass point at point $M$ to create circle $M$. Students should realize that circle

$M$ intersects circle $C$ at two points, $A$ and $B$. Using properties of circles and inscribed angles, students should be able to conclude that angle $PAC$ and angle $PBC$ are right angles. Since the line containing points $P$ and $A$ is perpendicular to a radius, $\overline{AC}$, of circle $C$, it is a tangent line passing through point $P$. Similarly, the line containing points $P$ and $B$ is also a tangent line.

- For expectations of this benchmark, constructions should be reasonably accurate and the emphasis is to make connections between the construction steps and the definitions, properties and theorems supporting them.
- While going over the steps of geometric constructions, ensure that students develop vocabulary to describe the steps precisely. *(MTR.4.1)*
- Problem types include identifying the next step of a construction, a missing step in a construction or the order of the steps in a construction.

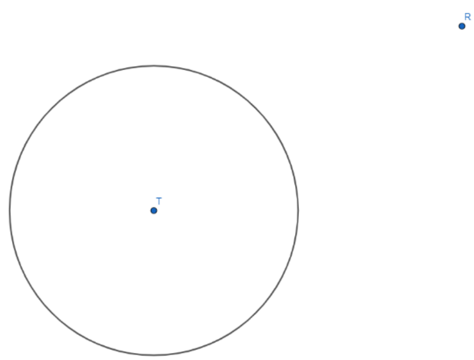
## Common Misconceptions or Errors

- Students may try to construct the tangent line by placing a straightedge from the external point to a point where it just touches the circle. To help address this, discuss with students how they could justify whether it will only touch the circle at one point without knowing which point that is.

## Instructional Tasks

*Instructional Task 1 (MTR.5.1)*

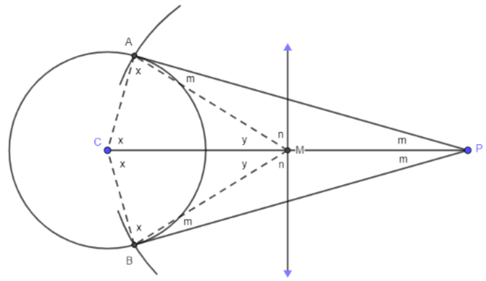
Circle $T$ and an external point $R$ are provided below.




Part A. Fill in the blank to complete the sentence below.
If a line is tangent to a circle it is ____________ to the radius of the circle at the point of tangency.
Part B. Draw segment $RT$.
Part C. Bisect segment $RT$ and label the midpoint $S$.
Part D. Construct circle $S$ using $ST$ as the length of the radius. In how many places does circle $S$ intersect circle $T$?
Part E. Label one of the points of intersection, as described in Part D, $V$. What is the measure of angle $TVR$?
Part F. What is the relationship between $\overline{TV}$ and line $RV$?
Part H. What can you conclude about line $RV$ and how it relates to circle $T$?

*Instructional Task 2 (MTR.4.1, MTR.5.1)*

A picture is given below that shows how Delia constructed a line tangent through point $P$ to circle $C$.

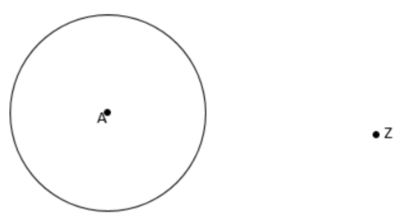



Part A. If circle $C$ and point $P$ were originally given, describe how Delia could have constructed points $M$, $B$ and $A$.
Part B. Classify triangle $AMC$, triangle $BMC$, triangle $AMP$ and triangle $BMP$.
Part C. Delia labeled certain angle measures as $x$, $m$, $y$ and $n$ indicating that certain pairs of angles are congruent. Justify her placement of these angle measures and their congruency.
Part D. Using the Triangle Sum Theorem two equations relating the angle measures for triangle $AMC$ and the angle measures for triangle $AMP$.
Part E. Use your equations from Part D to show that $2x + 2m + y + n = 360°$.
Part F. What do you notice, and can you conclude, about $y$ and $n$?
Part G. Using the equation from Part E and your conclusion from Part F, show that $x$ and $m$ are complementary angles.
Part H. How does your result from Part G show that Delia correctly constructed the tangent line, $\overline{AP}$, to circle $C$?

## Instructional Items

*Instructional Item 1*

Construct a line tangent to circle $A$ that goes through point $Z$.




**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

**MA.912.GR.6** *Use properties and theorems related to circles.*

*MA.912.GR.6.1*

## Benchmark

| | |
|---|---|
| **MA.912.GR.6.1** | Solve mathematical and real-world problems involving the length of a secant, tangent, segment or chord in a given circle. |

Benchmark Clarifications:
*Clarification 1:* Problems include relationships between two chords; two secants; a secant and a tangent; and the length of the tangent from a point to a circle.

## Connecting Benchmarks/Horizontal Alignment

## Terms from the K-12 Glossary

- MA.912.GR.3.2, MA.912.GR.3.3 MA.912.GR.5.3, MA.912.GR.5.4, MA.912.GR.5.5 MA.912.GR.7.2, MA.912.GR.7.3
- Circle

## Vertical Alignment

**Previous Benchmarks**

- MA.7.GR.1.3, MA.7.GR.1.4
- MA.8.GR.1.1
- MA.912.AR.1.2
- MA.912.AR.2.1
- MA.912.AR.3.1

**Next Benchmarks**

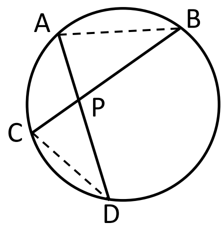
## Purpose and Instructional Strategies

In middle grades, students learned about circles, including the definition of center, radius (radii), diameter and circumference, and they used the Pythagorean Theorem to find lengths of segments. In Algebra 1, students rearranged formulas to highlight a quantity of interest and solved linear and quadratic equations in one variable. In Geometry, students learn about the lengths and relationships between a variety of line segments involving circles.

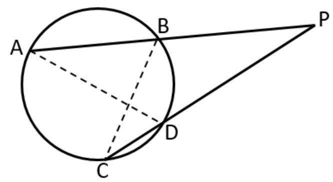
- Instruction includes using precise definitions and language when working with segments involving circles. Students should be able to determine the similarities and differences between each of the various segments and how their relationships interact with one another. (*MTR.4.1*)
  - For example, students should be able to answer questions like "What is the difference between chord and diameter?", "Is a diameter always a chord?" and "What is the difference between tangent and secant?"
- Instruction includes the understanding that while tangents and secants are often defined as lines, when determining lengths of these, one is referring to just a segment of the tangent or secant. The endpoints of these segments are typically a point of intersection between two lines, a point of intersection between a line and the circle, or a given point that is external to the circle. (*MTR.4.1*)
- Instruction includes the connection to similarity criteria and the Inscribed Angle Theorem to prove, and understand, the relationship between two chords. Typically, the theorem that describes the relationship between two lengths of chords is called the Intersecting Chords Theorem or the Chord-Chord Theorem.
  - For example, given the circle shown below with chords $\overline{AD}$ and $\overline{CB}$, two triangles are formed, $\Delta APB$ and $\Delta CPD$. Students should notice that angle $APB$ and angle $CPD$ are congruent because they are vertical angles. Students should also notice that angle $ABC$ and angle $ADC$ are congruent because they intercept the same arc on the circle. Therefore, using the Angle-Angle criterion, students can prove that these two triangles are similar. Since corresponding sides are proportional, students can determine that $\frac{AP}{CP} = \frac{BP}{DP}$. Using this fact, students can rearrange the formula to state that $AP \cdot DP = CP \cdot BP$, which is the Intersecting Chords

Theorem.

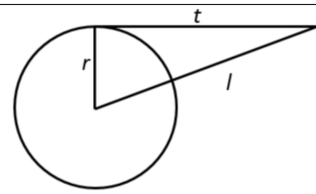
- Instruction includes the connection to similarity criteria and the Inscribed Angle Theorem to prove, and understand, the relationship between two secants. The Intersecting Secants Theorem, or the Secant-Secant Theorem, states that the product of the length of an entire secant segment and its external segment is equal to the product of the length of another




entire secant and its external segment.
  - For example, given the circle below, the Intersecting Secants Theorem states that $PA \cdot PB = PC \cdot PD$. This can be proved using Angle-Angle criterion to prove that $\Delta ADP \sim \Delta CBP$.




- Instruction includes relating the Intersecting Secants Theorem to the Tangent-Secant Theorem and the Tangent-Tangent Theorem. Students can build the understanding that the Tangent-Secant Theorem and the Tangent-Tangent Theorem are specific cases of the Intersecting Secants Theorem.
  - For example, to prove the Tangent-Secant Theorem, using the above circle, students can move points $C$ and $D$ towards one another until they meet at a tangent point, $T$, creating the tangent segment $TP$. Students then can determine that $PA \cdot PB = PT \cdot PT$ which is equivalent to $PA \cdot PB = PT^2$.
  - For example, to prove the Tangent-Tangent Theorem, using the above circle, students can move points $C$ and $D$ toward one another until they meet at a tangent point, $T$, creating the tangent segment $TP$. Likewise, students can move points $A$ and $B$ toward one another until they meet at a tangent point, $S$, creating the tangent segment $SP$. Students then can determine that $PS \cdot PS = PT \cdot PT$ which is equivalent to $PS = PT$.
- Instruction includes the connection to properties of perpendicular bisectors to prove, and understand, the relationship between a chord and the diameter of the circle that is perpendicular to the chord.
- Instruction includes the connection to the Pythagorean Theorem to prove, and understand, the relationship between a tangent segment, the radius of the circle to the point of tangency and the line segment from the center of the circle to the external point.
  - For example, given the circle below, students can use the Pythagorean Theorem to show that $t^2 + r^2 = l^2$. Students can then rearrange the formula to highlight the length of the tangent segment in terms of the length of the radius and the distance between the center and the external point.



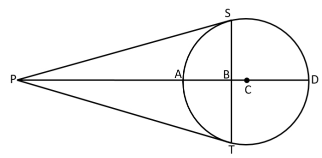

## Common Misconceptions or Errors

- Students may confuse the exterior portion of a secant and the whole secant when determining lengths or products of lengths.

## Instructional Tasks

*Instructional Task 1 (MTR.4.1, MTR.5.1)*

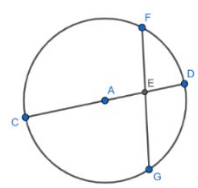
Circle $C$ is shown below with various lines and line segments. $\overline{PS}$ and $\overline{PT}$ are tangent to Circle $C$.




Part A. Write a statement of equality that involves the length of a tangent and the length of a secant.
Part B. Write a statement of equality that involves the lengths of two tangents.
Part C. Write a statement of equality based on the relationship between a chord and a diameter.
Part D. Write a statement of equality that involves the length of a tangent, the length of a line from the center to the external point and the length of a radius.
Part E. Compare your statements from Parts A, B, C and D with a partner.

*Instructional Task 2 (MTR.3.1)*

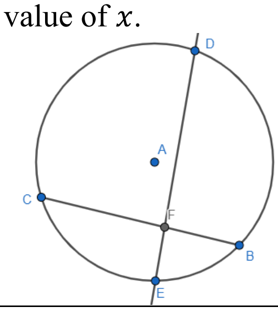
In Circle $A$, $AE = DE$, $FE = 6$ inches and $GE = 10$ inches. What is the length of the radius of Circle $A$?




## Instructional Items

*Instructional Item 1*

In Circle $A$, $\overline{DE}$ and $\overline{BC}$ intersect at point $F$. $FE = 1.3$ units, $BF = 1.9$ units, $FD = x + 1.3$ units and $CF = x$ units. Find the value of $x$.




**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

*MA.912.GR.6.2*

## Benchmark

**MA.912.GR.6.2** Solve mathematical and real-world problems involving the measures of arcs and related angles.

Benchmark Clarifications:
*Clarification 1:* Within the Geometry course, problems are limited to relationships between inscribed angles; central angles; and angles formed by the following intersections: a tangent and a secant through the center, two tangents, and a chord and its perpendicular bisector.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.3.2, MA.912.GR.3.3
- MA.912.GR.5.3, MA.912.GR.5.4, MA.912.GR.5.5
- MA.912.GR.7.2, MA.912.GR.7.3

## Terms from the K-12 Glossary

- Central Angle
- Circle
- Inscribed Angle

## Vertical Alignment

**Previous Benchmarks**

- MA.7.GR.1.3, MA.7.GR.1.4
- MA.7.DP.1.4
- MA.7.DP.2.3
- MA.912.AR.1.2
- MA.912.AR.2.1

**Next Benchmarks**

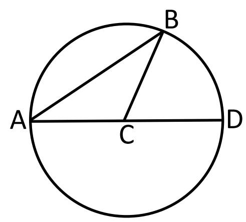
- MA.912.T.2.1, MA.912.T.2.2

## Purpose and Instructional Strategies

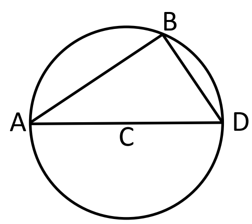
- In the middle grades, students encountered arc measure, arc length, and central angle by way of circle graphs and random experiments involving spinners, and they learned the formula for the circumference of a circle. In Algebra 1, students rearranged formulas to highlight a quantity of interest and solved linear equations in one variable. In Geometry, students learn a variety of results about angles and arcs in circles and how they relate to one another. In later courses, students will determine the value of trigonometric functions for real numbers by identifying angle measures in the unit circle. Instruction includes using precise definitions and language when working with angle measures and arcs involving circles. Students should be able to determine the similarities and differences between each of the various angle measures and arcs and how their relationships interact with one another. (*MTR.4.1*)
  - For example, students should be able to answer questions like "What is the difference between central angle and inscribed angle?", "Does the diameter have a central angle?" and "What is the difference between the measure of an inscribed angle and the measure of an intercepted arc?"
- Instruction includes student understanding of the Inscribed Angle Theorem and how it relates to angle measures within a circle. Proving this Theorem requires three cases: (1) the center of the circle is on one of the chords, so one of the chords is a diameter; (2) the

center of the circle is between the two chords; and (3) the center of the circle is not between the two chords. Below describes the proof of the first case of the Inscribed Angle Theorem.

- For example, given Circle $C$ with the chord $AB$ and the diameter $AD$, the central angle is $BCD$ and the inscribed angle is $DAB$, which is the same as $CAB$. Students should realize that $\measuredangle BCD + \measuredangle BCA = 180$. Students should also realize that $\measuredangle CAB + \measuredangle ABC + \measuredangle BCA = 180$. Therefore, after using the Substitution Property of Equality, $\measuredangle BCD + \measuredangle BCA = \measuredangle CAB + \measuredangle ABC + \measuredangle BCA$ which is equivalent to $\measuredangle BCD = \measuredangle CAB + \measuredangle ABC$. Since two of the sides of triangle ABC are radii, students can classify it as an isosceles triangle, therefore $\measuredangle CAB = \measuredangle ABC$. Students can use this fact to make the equivalent equation $\measuredangle BCD = \measuredangle CAB + \measuredangle CAB$ which is equivalent to $\measuredangle BCD = 2\measuredangle CAB$. This equation proves that the central angle, angle $BCD$, has twice the measure of the inscribed angle, angle $CAB$, which is the first case of the Inscribed Angle Theorem.




- Instruction includes the connection to coordinate geometry to prove, or justify, that every angle inscribed in a semicircle is a right angle.
  - For example, given Circle $C$ below with chords $AB$ and $BD$, students can use slope criteria to prove that angle $ABD$ is a right angle.




- Instruction includes student understanding of the following relationships between angle measures and arcs in a circle (even though some extend beyond *Clarification 1*). Students should understand that most relationships between angles in a circle, and relationships between segments in a circle, can be derived from the Inscribed Angle Theorem. *(MTR.5.1)*
  - The measure of a central angle is equal to the measure of its intercepted arc.
  - The measure of an inscribed angle is half the measure of its intercepted arc.
  - The measure of an inscribed angle is half the measure of the central angle that intercepts the same arc on the circle.
  - In a circle (or congruent circles), any two inscribed angles with the same intercepted arcs are congruent.
  - The measure of an angle formed by two tangents, two secants or a secant and a tangent from a point outside the circle, is half the difference of the measures of the intercepted arcs.
  - The angle made by two intersecting tangents to a circle is called a circumscribed

angle and it is supplementary to the central angle intercepting the same arc.
    - An angle formed by two intersecting chords and whose vertex is inside the circle equals one-half the sum of its intercepted arcs.
    - An angle formed by a chord and a tangent, whose vertex is on the circle, is one-half its intercepted arc.
    - If one chord of a circle is a perpendicular bisector of another chord, then the first chord is a diameter and bisects the arc intercepted by the chord.
- Instruction includes the understanding and naming of major, minor and semicircle arcs.
- Problem types include determining missing angle measures in both mathematical and real-world contexts.
- Instruction includes the connection to arc lengths in circles. (*MA.912.GR.6.4*)

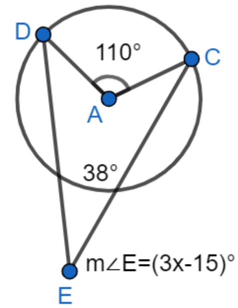
**Common Misconceptions or Errors**

- Students may try to apply properties of central angles and their intercepted arcs to inscribed angles and intercepted arcs.
- Students may confuse arc measure and arc length, and may try to measure arcs with linear units rather than degrees.

## Instructional Tasks

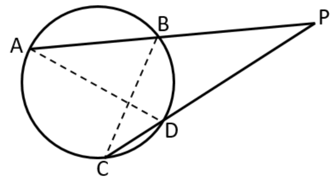
*Instructional Task 1 (MTR.3.1)*

Find the measure of angle $E$ in circle $A$.




*Instructional Task 2 (MTR.2.1, MTR.4.1, MTR.5.1)*

A circle is given below with two intersecting secants, $\overline{PA}$ and $\overline{PC}$.




Part A. What is the sum of the measures of angle $BCP$, angle $CPB$ and angle $PBC$?

Part B. What is the sum of the measures of angle $PBC$ and angle $ABC$?

Part C. What can you conclude about the relationship between the sum of the measures of the three angles from Part A and the sum of the measures of the two angles from Part B?

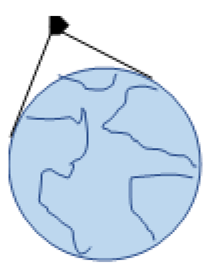
Part D. Using the information from Part C, what can you conclude about the measure of angle $CPB$? State your conclusion algebraically as an equation where $\measuredangle CPB =$?.

Part E. How can you use the information from Part D, to justify the Secant-Secant Angle Theorem which states that $\measuredangle CPB = \frac{m\widehat{AC} - m\widehat{BD}}{2}$?

## Instructional Items

*Instructional Item 1*

The International Space Station (ISS) passes over the earth 248 miles above the earth's surface. The angle formed between the two tangents formed from the ISS and the earth measures 140.4˚. What is the measure of the arc of the earth that could have a view of the ISS passing overhead?



*The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

MA.912.GR.6.3

## Benchmark

**MA.912.GR.6.3** Solve mathematical problems involving triangles and quadrilaterals inscribed in a circle.

Benchmark Clarifications:
*Clarification 1:* Instruction includes cases in which a triangle inscribed in a circle has a side that is the diameter.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.3.2, MA.912.GR.3.3
- MA.912.GR.5.3, MA.912.GR.5.4
- MA.912.GR.7.2, MA.912.GR.7.3

## Terms from the K-12 Glossary

- Circle
- Circumscribed Circle
- Diameter
- Inscribed Polygon in a Circle
- Quadrilateral
- Triangle

## Vertical Alignment

**Previous Benchmarks**

- MA.8.GR.1.6
- MA.912.AR.1.2
- MA.912.AR.2.1

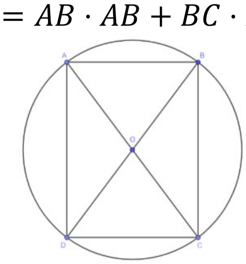
**Next Benchmarks**

## Purpose and Instructional Strategies

In grade 8, students learned about regular polygons and their interior angles. In Algebra 1, students rearranged formulas to highlight a quantity of interest and solved linear equations in one variable. In Geometry, students learn about the properties of the interior angles of triangles and quadrilaterals that are inscribed in a circle and how they relate to one another.

- Instruction includes using precise definitions and language when working with triangles and quadrilaterals inscribed in a circle. Students should be able to determine the similarities and differences between each and how their relationships interact with one another. (*MTR.4.1*)
  - For example, students should be able to answer questions like "What is the difference between a polygon inscribed in a circle and a polygon circumscribed about a circle?" and "What is the difference between a circle inscribed in a polygon and a polygon inscribed in a circle?"
- Instruction includes the connection to constructing equilateral triangles and squares inscribed in a circle.
- Instruction includes exploring various triangles inscribed in a circle.
  - For example, when one side of a triangle inscribed in a circle is the diameter of the circle, then the triangle is inscribed in a semicircle. Students should determine what kind of angles are the interior angles of the triangle inscribed in a semicircle and should be able to identify them as inscribed angles. Students can make the connection to the angle opposite the diameter divides the circle into two equal parts, therefore the inscribed angle is one-half of 180°. This conclusion can lead students to Thales' Theorem which states that the angle inscribed in a semicircle is always a right angle. That is, if $A$, $B$ and $C$ are distinct points on a circle where $\overline{AC}$ is a diameter, the angle $ABC$ is a right angle. Therefore, the triangle $ABC$ is always a right triangle.

- Instruction includes the connection to the relationships between inscribed angles and their intercepted arcs, when a triangle is inscribed in a circle, and the Triangle Sum Theorem. When a triangle is inscribed in a circle, the three arcs intercepted by the interior angles complete a circle, so the sum of the measures of the interior angles of the triangle is half the measure of a circle in degrees, 180˚. *(MTR.5.1)*
- A quadrilateral inscribed in a circle is also called a cyclic quadrilateral. The opposite angles of such quadrilateral are supplementary, and each exterior angle is equal to the interior opposite angle. To prove the first of these properties, students can determine what type of angles are the interior angles of the quadrilateral inscribed in a circle. Then students can discover that opposite angles of the quadrilateral have intercepted arcs such that they complete a circle. Using the relationship between inscribed angles and their intercepted arcs, students should be able to conclude that the opposite angles of cyclic quadrilaterals are supplementary.
- Instruction includes the understanding that there are two definitions of an inscribed circle in a polygon. One definition states that the inscribed circle is the largest circle that fits inside the polygon. Under this definition, all polygons have inscribed circles, with some polygons (e.g., rectangles) having more than one inscribed circle. The other definition states that the inscribed circle is the circle that fits inside of the polygon and touches all of its sides. Under this definition, only special polygons (e.g., regular polygons) have inscribed circle. Students can explore the differences by constructing an inscribed circle in a quadrilateral.
  - For example, given a quadrilateral that is not a rectangle, students can extend two opposite sides to form a triangle and construct the inscribed circle of that triangle. Students can repeat this for the other two opposite sides of the quadrilateral. Students should be able to reason that one of the two constructed circles is the largest circle that fits inside the given quadrilateral.
- Extension of this benchmark includes the introduction of Ptolemy's Theorem with cyclic quadrilaterals. This theorem states that the product of the diagonals is equal to the sum of the product of its two pairs of opposite sides. Students should realize that when this theorem is applied to a rectangle inscribed in a circle, it can be used to prove the Pythagorean Theorem.
  - For example, given rectangle $ABCD$ inscribed in circle $O$. Students can apply Ptolemy's Theorem, so that $AC \cdot BD = AB \cdot CD + AD \cdot BC$. Since the opposite sides of a rectangle are congruent and the diagonals of a rectangle are congruent, then $AB = CD$, $AD = BC$ and $AC = BD$. Using the Substitution Property of Equality, then $AC \cdot AC = AB \cdot AB + BC \cdot BC$. That is $AC^2 = AB^2 + BC^2$.



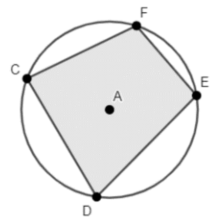

## Common Misconceptions or Errors

- Students may think that every quadrilateral can be inscribed in a circle. To help address this misconception, have students use a triangle inscribed in a circle to see that there are many places a fourth vertex can be placed to form a quadrilateral that cannot be inscribed in the circle.

## Instructional Tasks

*Instructional Task 1 (MTR.3.1)*

Quadrilateral $DCFE$ is inscribed in Circle $A$ and the measure of angle $D$ is $75^\circ$.




Part A. What is the measure of angle $CAE$? What is the measure of angle $CFE$?
Part B. What is the measure of arc $CDE$?
Part C. What can you determine about the measures of angle $DCF$ and angle $FED$?

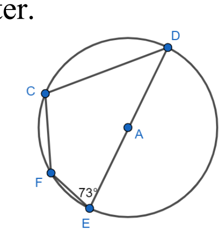
*Instructional Task 2 (MTR.4.1)*

Given a quadrilateral is inscribed in a circle and one of the diagonals is a diameter of the circle. Classify the possible types of quadrilaterals it could be.

## Instructional Items

*Instructional Item 1*

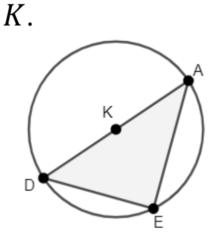
In circle $A$, segment $DE$ is a diameter.




Part A. Determine the measure of angle $C$.
Part B. If the measure of arc $CF$ is $50^\circ$, determine the measures of angle $D$ and angle $F$.

*Instructional Item 2*

Triangle $DAE$ is inscribed in Circle $K$.




Part A. Determine the value of $x$ if the measure of angle $E$ is $(2x + 30)^\circ$.
Part B. Determine the measure of angle $D$ if the measure of angle $A$ is $(2x - 20)^\circ$.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

*MA.912.GR.6.4*

## Benchmark

**MA.912.GR.6.4** Solve mathematical and real-world problems involving the arc length and area of a sector in a given circle.

Benchmark Clarifications:
*Clarification 1:* Instruction focuses on the conceptual understanding that for a given angle measure the length of the intercepted arc is proportional to the radius, and for a given radius the length of the intercepted arc is proportional is the angle measure.

### Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.5.3, MA.912.GR.5.4, MA.912.GR.5.5
- MA.912.GR.7.2, MA.912.GR.7.3

### Terms from the K-12 Glossary

- Circle
- Radius

### Vertical Alignment

**Previous Benchmarks**

- MA.7.GR.1.3, MA.7.GR.1.4
- MA.912.AR.1.2
- MA.912.AR.2.1

**Next Benchmarks**

- MA.912.T.2.1, MA.912.T.2.2

### Purpose and Instructional Strategies

In grade 7, students solved problems involving the circumference and area of a circle. In Algebra 1, students rearranged formulas to highlight a quantity of interest and solved linear equations in one variable. In Geometry, students use their knowledge of circumference and area to relate arc measure to arc length and to area of sectors of circles. In later courses, students will determine the value of trigonometric functions for real numbers by identifying angle measures in the unit circle, and will convert between radians and degrees.

- Instruction includes the student understanding that arcs can be measured in both degrees (arc measure) and in units of length (arc length). For expectations of Geometry, students will only need to work in degrees when discussing arc measure as students will work with radians in later courses.
- Instruction includes the understanding that two or more circles with a common center, called concentric circles, will have the same arc measure but different corresponding arc lengths.
  - For example, given various concentric circles, students can draw a central angle and extend its sides to the length of the radius of the largest circle. Students should notice that the measure of the central angle remains the same and that the larger the circle; the longer the intercepted arc; and that the length of the intercept arc depends proportionally on the radius in the same way that the circumference depends proportionally on the radius.
- When determining an arc length or an area of a sector given the arc measure, instruction includes the connection to proportional relationships (as was done in grade 7).
  - For example, if the arc measure is 57°, and students are asked to find the area of the sector, they can determine the area of the entire circle and multiply by $\frac{57}{360}$. Students should realize that areas of sectors are fractional portions of the area of the entire circle.
  - For example, if the arc measure is 57°, and students are asked to find the arc length, they can determine the circumference of the entire circle and multiply by $\frac{57}{360}$.

Students should realize that arc lengths are fractional portions of the circumference of the entire circle.

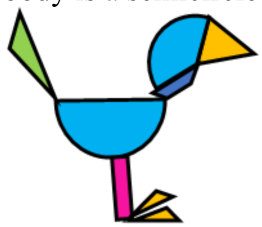
## Common Misconceptions or Errors

- Students may confuse arc measure and arc length, and may try to measure arcs with linear units rather than degrees.

## Instructional Tasks

*Instructional Task 1 (MTR.7.1)*

De'Veon must create an animal using geometric shapes for his Geometry class. He has decided to use construction paper scraps from his mom's crafting box to create a bird, like the one shown below. The head is a made from a sector with radius 1.5 centimeters and central angle measuring 130˚. The body is a semicircle with radius 1.9 centimeters.



Part A. What fraction of the whole circle is the head?

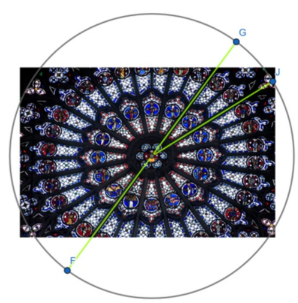
Part B. How much glitter string will he need to outline the part of the bird's head that is not touching the beak or neck?

Part C. What is the total area of light blue construction paper used to create the bird (i.e., the area of the head and the body)?

## Instructional Items

*Instructional Item 1*

The North Rose Window in the Rouen Cathedral in France has a diameter of 23 feet. The stained glass design is equally spaced about the center of the circle. What is the area of the sector bounded by arc $GJ$?




**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

# MA.912.GR.6.5

## Benchmark

**MA.912.GR.6.5** Apply transformations to prove that all circles are similar.

## Connecting Benchmarks/Horizontal Alignment

## Terms from the K-12 Glossary

- MA.912.GR.2.1, MA.912.GR.2.2, MA.912.GR.2.3, MA.912.GR.2.5, MA.912.GR.2.8
- MA.912.GR.7.2, MA.912.GR.7.3

- Circle
- Dilation
- Similarity
- Translation

## Vertical Alignment

**Previous Benchmarks**

- MA.8.GR.2.1, MA.8.GR.2.2

**Next Benchmarks**

## Purpose and Instructional Strategies

In grade 8, students learned about similarity and similarity transformations. In Geometry, students apply transformations to prove that all circles are similar.

- Instruction includes presenting students with a pair of circles of different size and asking them to identify a sequence of transformations that would map one onto the other. Students should realize that a single translation and a single dilation is all that is needed in a sequence to map one onto the other.
- Instruction includes the connection to the coordinate plane by showing that two circles are similar using coordinates.
- Students should connect the definition of similarity in terms of corresponding parts applied to polygons and explore what parts of the circles will be in proportion between the preimage and image of a dilations.
    - For example, given two circles, their radii ($r_1$ and $r_2$) and their diameters ($d_1$ and $d_2$) would satisfy the proportional relationship $\frac{r_1}{r_2} = \frac{d_1}{d_2}$.

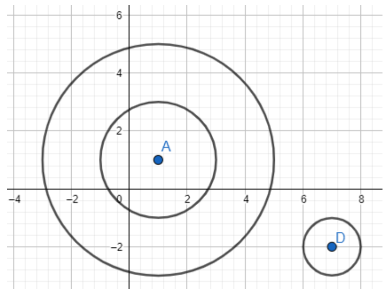
## Common Misconceptions or Errors

- Students may think that always need a formal proof to prove that all circles are similar.

## Instructional Tasks

*Instructional Task 1 (MTR.4.1)*

Two concentric circles with point $A$ as the center and circle $B$ are given on the coordinate plane.



Part A. Describe the transformation(s) needed to map the smaller circle $A$ onto the larger circle $A$.

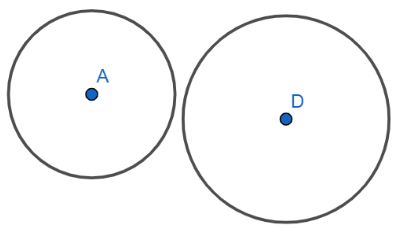
Part B. List the transformation(s) that could be used to show that each circle $A$ is similar to circle $D$. Compare your transformations with a partner.

Part C. What is the difference in the transformation(s) depending on the circle $A$ chosen?

## Instructional Items

*Instructional Item 1*

Circle $A$ and circle $D$ are given below.




Part A. Describe a set of transformations that could be used on circle $A$ to show it is similar to circle $D$.

Part B. Describe a set of transformations that could be used on circle $D$ to show it is similar to circle $A$.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

**MA.912.GR.7** *Apply geometric and algebraic representations of conic sections.*

### *MA.912.GR.7.2*

## Benchmark

| | |
|---|---|
| **MA.912.GR.7.2** | Given a mathematical or real-world context, derive and create the equation of a circle using key features. |

Benchmark Clarifications:
*Clarification 1:* Instruction includes using the Pythagorean Theorem and completing the square.
*Clarification 2:* Within the Geometry course, key features are limited to the radius, diameter and the center.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.3.3
- MA.912.GR.5.3, MA.912.GR.5.4, MA.912.GR.5.5
- MA.912.GR.6

## Terms from the K-12 Glossary

- Circle
- Diameter
- Radius

## Vertical Alignment

**Previous Benchmarks**

- MA.8.GR.1.1, MA.8.GR.1.2
- MA.912.AR.1.2
- MA.912.AR.3.7
- MA.912.F.2

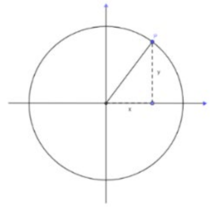
**Next Benchmarks**

- MA.912.GR.7

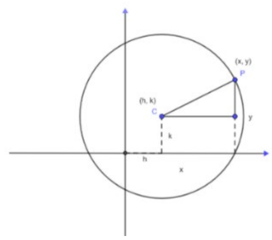
## Purpose and Instructional Strategies

In grade 8, students solved problems involving the Pythagorean Theorem. In Algebra 1, students rearranged formulas to highlight a quantity of interest, worked with quadratic equations and functions and performed single transformations on functions. In Geometry, this background provides a solid foundation for writing the equation of a circle to represent a variety of contexts, based on key features of the circle. In later courses, students will expand these skills to write equations for other circles, parabolas, hyperbolas and ellipses based on their key features.

- Instruction includes students deriving the equation of a circle centered at the origin and at a point $(h, k)$. Students should make the connection to the Pythagorean Theorem by drawing right triangles from the center of the circle. Students should make the connection between the equations centered at the origin and at a point $(h, k)$; realizing that if the point $(h, k)$ is the origin then the equation can be written as either $x^2 + y^2 = r^2$ or as $(x - 0)^2 + (y - 0)^2 = r^2$.
  - For example, given a circle centered at the origin on the coordinate plane and have students select a point $P(x, y)$ on the circle. Students should be able to see that the distance from $P$ to the center is the radius of the circle. Students can then draw a right triangle, like the one shown below, that includes the center and $P$ as two of its vertices. Students can use their knowledge of the Pythagorean Theorem to develop the equation $x^2 + y^2 = r^2$, showing the relationship between $x$, $y$ and the radius, $r$. Students should be able to realize that this equation is valid for every point $P(x, y)$ on the circle and is not valid for any point not on the circle.



- For example, given a circle centered at $C(h, k)$ on the coordinate plane and have students select a point $P(x, y)$ on the circle. Students should be able to see that the distance from $P$ to the center is the radius of the circle. Students can then draw a right triangle, like the one shown below, that includes the center and $P$ as two of its vertices. Students can use their knowledge of the Pythagorean Theorem to develop the equation $(x-h)^2+(y-k)^2=r^2$, showing the relationship between $x$, $y$, $h$, $k$ and the radius, $r$. Students should be able to realize that this equation is valid for every point $P(x, y)$ on the circle and is not valid for any point not on the circle.



- Instruction includes the connection to transformation of functions (as was done in Algebra 1) when developing an understanding of the relationship between the equations $x^2+y^2=r^2$ and $(x-h)^2+(y-k)^2=r^2$.
- Students should have experience working with the equation of a circle in center-radius form, $(x-h)^2+(y-k)^2=r^2$, and in general form, $Ax^2+By^2+Cx+Dy+E=0$ where $A$ and $B$ are the same real, non-zero number. Students may need to convert from one form to another, particularly from the general form to the center-radius form. Instruction includes discussing how this conversion can be done and the benefits of each form of the equation of a circle.
- Instruction includes exploring the pattern of completing the square for the general form and the way the center and the radius can be obtained from this procedure. Exploration includes cases where the radius may result in a complex number; noting that the equation does not give a circle when this happens.
  - For example, given the equation $Ax^2+By^2+Cx+Dy+E=0$ when $A=1$ and $B=1$, students can rewrite this as $x^2+Cx+y^2+Dy=-E$. Then students can complete the square for both the $x$-part of the equation and the $y$-part of the equation. Students should obtain the equation $\left(x+\frac{C}{2}\right)^2+\left(y+\frac{D}{2}\right)^2=-E+\left(\frac{C}{2}\right)^2+\left(\frac{D}{2}\right)^2$. Students should then compare this to center-radius form to determine that the center of the circle is at the point $\left(-\frac{C}{2},-\frac{D}{2}\right)$ and the radius is $\sqrt{-E+\left(\frac{C}{2}\right)^2+\left(\frac{D}{2}\right)^2}$.
- Problem types include cases where $A$ and $B$ are equivalent, but not equal to 1.

- For enrichment of this benchmark, instruction includes the connection to finding the equation of the circumscribed circle about a triangle whose vertices are given on the coordinate plane. *(MTR.5.1)*

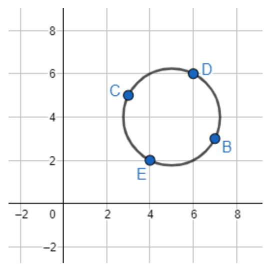
## Common Misconceptions or Errors

- Students may confuse the signs of $(h, k)$ when determining the center of the circle.
  - For example, given the equation $(x-1)^2 + (y+2)^2 = 1$, the center is $(1, -2)$ and not $(-1, 2)$.
- Students may forget to take the square root of the $r^2$ term to determine the radius of the circle.

## Instructional Tasks

*Instructional Task 1 (MTR.4.1)*

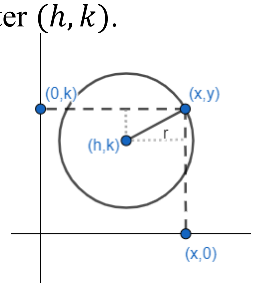
A circle on the coordinate plane is given. Segments $CB$ and $ED$ are diameters of circle $A$. Point $C$ is located at $(3,5)$, point $D$ is located at $(6,6)$, point $B$ is located at $(7,3)$ and point $E$ is located at $(4,2)$.




Part A. Determine the center of the circle. Explain your method.
Part B. Find the length of the radius of circle $A$. Explain your method.
Part C. Write the equation of the circle and check that the points $B$, $C$, $D$ and $E$ satisfy the equation.

*Instructional Task 2 (MTR.2.1, MTR.5.1)*

Point $(x, y)$ is on a circle with center $(h, k)$.




Part A. The horizontal distance between point $(x, y)$ and center $(h, k)$ can be represented as _____. The vertical distance between the point $(x, y)$ and center $(h, k)$ can be represented as _____.
Part B. Using Pythagorean Theorem, write an equation for the radius in terms of $(x, y)$ and $(h, k)$.
Part C. Because $(x, y)$ could be any point on the circle, this is the equation of a circle where $(h, k)$ is the ______________ of the circle and $r$ is the _____________ of the circle.

*Instructional Task 3 (MTR.7.1)*

A school's campus is designed in the shape of a circle. The architect would like to place the cafeteria equidistant from the Freshman building and from the Senior building. On a coordinate plane, the Freshman building is located at the point $(431, 219)$ and the Senior building is located at the point $(0,0)$, where the coordinates are given in feet. Assume that the endpoints of the diameter of circle are the Freshman building and the Senior building and that the cafeteria is on the line connecting the two buildings.

Part A. Determine the location of the cafeteria.
Part B. How far is it from the cafeteria to the Freshman building?
Part C. Write an equation that represents the boundary of the circular campus.
Part D. If the campus were to have a circular fence along its boundary, what is the total length of the fence, in feet?

## Instructional Items

*Instructional Item 1*

Given the equation $x^2 + 2x + y^2 - 4y + E = 0$, determine possible values of $E$ such that the equation is an equation of a circle.

*Instructional Item 2*

What is the equation of a circle centered at $(-1, 2)$, with a diameter of 2 units?

*Instructional Item 3*

What is the equation of the circle centered at $(-2, -5)$ and passing through $(5, 0)$?

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

*MA.912.GR.7.3*

## Benchmark

| | |
|---|---|
| **MA.912.GR.7.3** | Graph and solve mathematical and real-world problems that are modeled with an equation of a circle. Determine and interpret key features in terms of the context. |

Benchmark Clarifications:
*Clarification 1:* Key features are limited to domain, range, eccentricity, center and radius.
*Clarification 2:* Instruction includes representing the domain and range with inequality notation, interval notation or set-builder notation.
*Clarification 3:* Within the Geometry course, notations for domain and range are limited to inequality and set-builder.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.3.3
- MA.912.GR.5.3, MA.912.GR.5.4, MA.912.GR.5.5
- MA.912.GR.6

## Terms from the K-12 Glossary

- Circle
- Domain
- Radius
- Range of a Relation or Function

## Vertical Alignment

**Previous Benchmarks**

**Next Benchmarks**

- MA.8.GR.1.1, MA.8.GR.1.2
- MA.912.AR.3.8
- MA.912.F.2
- MA.912.GR.7

## Purpose and Instructional Strategies

In grade 8, students solved problems involving the Pythagorean Theorem. In Algebra 1, students rearranged formulas to highlight a quantity of interest, worked with quadratic equations and functions and performed single transformations on functions. In Geometry, this background provides a solid foundation for graphing the equation of a circle and determining and interpreting key features. In later courses, students will expand these skills to graph equations for other circles, parabolas, hyperbolas and ellipses and to determine and interpret their key features.

- Instruction includes various methods for students to graph the equation of a circle.
  - For example, students can use the equation of a circle to identify and plot the center of the circle on a coordinate plane. Then students can identify and use the radius to determine at least 4 points on the circle and plot those points. Students can then use those 4 points to sketch the graph of the circle, or use a compass to graph the circle.
- Instruction includes identifying and interpreting the domain and the range of the equation of a circle. Students should understand that these are the domain and the range of a relation, not a function.
  - For example, the domain and the range can be determined by the coordinates of the center $(h, k)$, and the radius, $r$. The domain is $h - r \leq x \leq h + r$ and the range is $k - r \leq y \leq k + r$. The domain can be interpreted as the points on the circle having a minimum $x$-value at $h - r$ and a maximum $x$-value at $h + r$. Likewise, range can be interpreted as the points on the circle having a minimum $y$-value at $k - r$ and a maximum $y$-value at $k + r$.
- Within the Geometry course, it is not an expectation for students to master the concept of eccentricity as a key feature of circles even though it is mentioned within *Clarification 1*. For enrichment purposes, eccentricity can be included within instruction; the eccentricity of all circles is zero.
- Problem types include graphing circles, determining key features of circles and interpreting key features of circles within a mathematical or real-world context.

## Common Misconceptions or Errors

- Students may try to always plot the center of the circle at the origin, not the actual center of the circle.
- Students may confuse the signs of $(h, k)$ when determining the center of the circle.
  - For example, given the equation $(x - 1)^2 + (y + 2)^2 = 1$, the center is $(1, -2)$ and not $(-1, 2)$.
- Students may forget to take the square root of the $r^2$ term to determine the radius of the circle.

## Instructional Tasks

*Instructional Task 1 (MTR.4.1, MTR.7.1)*

Nikita is trying to determine which sprinkler to buy for her backyard. One rotating sprinkler has a throwing radius of 32 feet, which costs \$13.99, and the other rotating sprinkler has a throwing radius of 42 feet, which costs \$16.99. Note that the sprinkler throwing radius refers to the radius of the spray when the sprinkler is being used.

Part A. Write an equation that describes the region each sprinkler will cover if centered at the position $(h, k)$.

Part B. Nikita's backyard is approximately a rectangle with dimensions 80 feet by 110 feet. Nikita would like to place her sprinklers so that she waters the majority of her backyard, doubling coverage with two or more sprinklers when necessary. Develop a pattern of sprinklers that would cover the backyard.

Part C. Compare your sprinkler pattern and cost with a partner. Can you and your partner determine a better, and cheaper, solution?

## Instructional Items

*Instructional Item 1*

A florist serving the Orlando area located at (9,8), and marked with an X on the coordinate plane shown where each unit is 10 miles. The florist has a 50-mile delivery radius.


Part A. Write an equation that describes the delivery area.

Part B. Does any of the florist's delivery area include part of Seminole County?

*Instructional Item 2*

The equation of a circle is given.

$$x^2 + y^2 - 6x + 8y + 5 = 0$$

Part A. Determine the center and the radius of the circle.

Part B. What is the ordered pair that contains the maximum $y$-value of the circle?

Part C. Sketch the graph of the circle on the coordinate plane.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

## Trigonometry

**MA.912.T.1** *Define and use trigonometric ratios, identities or functions to solve problems.*

### *MA.912.T.1.1*

**Benchmark**

**MA.912.T.1.1** Define trigonometric ratios for acute angles in right triangles.

Benchmark Clarifications:
*Clarification 1:* Instruction includes using the Pythagorean Theorem and using similar triangles to demonstrate that trigonometric ratios stay the same for similar right triangles.
*Clarification 2:* Within the Geometry course, instruction includes using the coordinate plane to make connections to the unit circle.
*Clarification 3:* Within the Geometry course, trigonometric ratios are limited to sine, cosine and tangent.

**Connecting Benchmarks/Horizontal Alignment**

- MA.912.GR.1.1, MA.912.GR.1.2, MA.912.GR.1.6
- MA.912.GR.7.2, MA.912.GR.7.3

**Terms from the K-12 Glossary**

- Angle
- Hypotenuse
- Right Triangle

**Vertical Alignment**

**Previous Benchmarks**

- MA.8.GR.1.1
- MA.8.GR.2.4

**Next Benchmarks**

- MA.912.T.1.5, MA.912.T.1.6, MA.912.T.1.7, MA.912.T.1.8
- MA.912.T.2
- MA.912.T.3

**Purpose and Instructional Strategies**

In grade 8, students solved problems involving similar triangles. In Geometry, students will use similarity in right triangles to develop the definition of trigonometric ratios. In later courses, trigonometric ratios will be used to define trigonometric functions on the unit circle and the real line.

- Instruction includes students developing the definitions for sine, cosine and tangent using ratios of side lengths in similar triangles and the Pythagorean Theorem.
  - For example, given a set of similar right triangles each with one side length missing, students can find the missing side lengths using the Pythagorean Theorem. Additionally, students can determine the constant of proportionality among the similar triangles. Students can then calculate the trigonometric ratios for each triangle and conclude that the corresponding ratios are the same for each triangle. Lead students to conclude the trigonometric ratios correspond to the angle measure and that each angle measure of an acute angle can be assigned to a sine, cosine, and tangent using an appropriate right triangle.
- Instruction includes discussing the relationships between the sine, cosine and tangent of the two acute angles of a triangle.
  - For example, given a right triangle $ABC$, where angle $C$ is the right angle, students can determine that $\sin A = \cos B$, $\cos A = \sin B$ and $\tan A = \frac{1}{\tan B}$. Students should

realize that these equations are true as long as angle $A$ and angle $B$ are complementary angles.

- Instruction includes the use of technology, including a calculator, to show how the sine, cosine and tangent of any acute angle (and more) can be found.
- Within the Geometry course, the expectation is to use angle measures given in degrees and not in radians. Additionally, it is not the expectation for students to master the trigonometric ratios of secant, cosecant and cotangent within this course.
- It is customary to use Greek letters to represent angle measures (e.g., $\theta, \alpha, \beta, \gamma$).
- Instruction includes the connection to the unit circle.
    - For example, given a right triangle on the coordinate plane with vertices at $(0,0)$, $(x, 0)$ and $(x, y)$, and the length of the hypotenuse equal to 1. Students can find the sine, cosine and tangent of the acute angle that the hypotenuse forms with the $x$-axis.

## Common Misconceptions or Errors

- Students may misidentify the sides of triangles.
    - For example, students may identify the hypotenuse as being the adjacent leg or confuse the adjacent and opposite sides.

## Instructional Tasks

*Instructional Task 1 (MTR.2.1)*

Provide students with a set of similar right triangles and their side lengths.

Part A. Identify corresponding angles and label them as $\angle A$, $\angle B$ and $\angle C$, with $\angle C$ denoting the right angle.

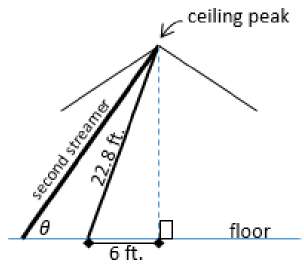
Part B. Write the following ratios for each one of the right triangles with respect to $\angle A$: $opposite\ leg{:}\ hypotenuse$, $adjacent\ leg{:}\ hypotenuse$ and $opposite\ leg{:}\ adjacent\ leg$. What do you notice?

Part C. How do your ratios created in Part B relate to sine, cosine and tangent?

## Instructional Items

*Instructional Item 1*

Belle is hanging streamers for her brother's surprise birthday party. She secures two streamers of different lengths at the peak of the ceiling. The center of the floor is directly underneath the ceiling peak. The distance along the floor from the center of the room to where the first streamer is attached is 6 feet. The second streamer is attached to the floor further from the center of the floor than the first streamer.




The distance between the streamers is $x$ feet and the length of the second streamer is $y$ feet. The angle formed between the second streamer and the floor is $\theta$. Select all of the equations that are true to the nearest tenth based on the diagram.

a. $\sin\theta = \frac{22.0}{y}$

b. $\sin\theta = \frac{22.8}{y}$

c. $\tan\theta = \frac{22.0}{6}$

d. $\cos\theta = \frac{x}{y}$

e. $\cos\theta = \frac{x+6}{22.8}$

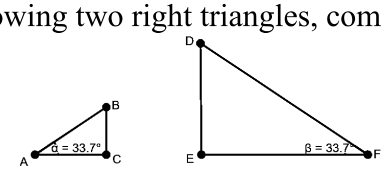
f. $\tan\theta = \frac{22.0}{x+6}$

g. $\sin\theta = \frac{22.0}{22.8}$

h. $\tan\theta = \frac{22.8}{x}$

*Instructional Item 2*

Given the diagram below showing two right triangles, complete the following statements.




Statement A. $\sin 33.7° = \frac{BC}{\square}$

Statement B. $\sin 33.7° = \frac{\square}{DF}$

Statement C. $\frac{BC}{AC} = \frac{\square}{\square}$

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

*MA.912.T.1.2*

## Benchmark

| | |
|---|---|
| **MA.912.T.1.2** | Solve mathematical and real-world problems involving right triangles using trigonometric ratios and the Pythagorean Theorem. |

Benchmark Clarifications:
*Clarification 1:* Instruction includes procedural fluency with the relationships of side lengths in special right triangles having angle measures of $30° - 60° - 90°$ and $45° - 45° - 90°$.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.1.2, MA.912.GR.1.6
- MA.912.GR.6.3

## Terms from the K-12 Glossary

- Angle
- Equilateral Triangle
- Hypotenuse
- Isosceles Triangle
- Right Triangle

## Vertical Alignment

**Previous Benchmarks**

- MA.8.GR.1.1, MA.8.GR.1.2

**Next Benchmarks**

- MA.912.T.1.5, MA.912.T.1.6, MA.912.T.1.7, MA.912.T.1.8
- MA.912.T.2
- MA.912.T.3

## Purpose and Instructional Strategies

In grade 8, students solved problems involving right triangles using the Pythagorean Theorem. In Geometry, students use their understanding of sine, cosine and tangent to solve mathematical and real-world problems involving right triangles. In later courses, students will extend this knowledge to solve more difficult problems with right triangles, and extend the concept of trigonometric ratios to trigonometric functions on the unit circle and the number line.

- Within the Geometry course, the expectation is to use angle measures given in degrees and not in radians. Additionally, it is not the expectation for students to master the trigonometric ratios of secant, cosecant and cotangent within this course.
- It is customary to use Greek letters to represent angle measures (e.g., $\theta, \alpha, \beta, \gamma$).
- Problem types include cases where some information about the side lengths or angle measures of a right triangle is missing and one must use trigonometric ratios, inverse of trigonometric ratios or Pythagorean Theorem to determine the unknown length(s) or angle measure(s) within a mathematical or real-world context.
- Instruction includes the concept of inverse trigonometric ratios to determine unknown angle measures and how to find these values using technology, including a calculator. Students should have practice using both notations for the inverse trigonometric ratios ($\sin^{-1} A$ or $\arcsin A$; $\cos^{-1} A$ or $\arccos A$; and $\tan^{-1} A$ or $\arctan A$).
- Instruction includes exploring the relationships of the side lengths of special right triangles $45° - 45° - 90°$ and $30° - 60° - 90°$.
  - For example, students should realize that the special right triangle $45° - 45° - 90°$ is an isosceles right triangle. Therefore, two of its angle measures and side lengths are equivalent. So, if a side length is $x$ units, then students can use the

Pythagorean Theorem to determine that the hypotenuse is $x\sqrt{2}$ units. Additionally, students can make the connection to its trigonometric ratios: $\sin 45° = \frac{1}{\sqrt{2}}$ (or equivalently $\frac{\sqrt{2}}{2}$); $\cos 45° = \frac{1}{\sqrt{2}}$ (or equivalently $\frac{\sqrt{2}}{2}$); and $\tan 45° = 1$.

  - For example, students should realize that the special right triangle $30° - 60° - 90°$ is half of an equilateral triangle. Students can use that knowledge to determine that the shorter leg is one-half the length of the hypotenuse. So, if the shorter leg is $x$ units and the hypotenuse is $2x$ units, then students can use the Pythagorean Theorem to determine that the other leg is $x\sqrt{3}$ units. Additionally, students can make the connection to its trigonometric ratios such as, $\sin 30° = \frac{1}{2}$; $\cos 30° = \frac{\sqrt{3}}{2}$ and $\tan 30° = \frac{1}{\sqrt{3}}$ (or equivalently $\frac{\sqrt{3}}{3}$).

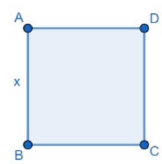
## Common Misconceptions or Errors

- Students may choose the incorrect trigonometric ratio when solving problems.
- Students may misidentify the sides of triangles.
  - For example, students may identify the hypotenuse as being the adjacent leg or confuse the adjacent and opposite sides.

## Instructional Tasks

*Instructional Task 1 (MTR.3.1)*

$ABCD$ is a square.

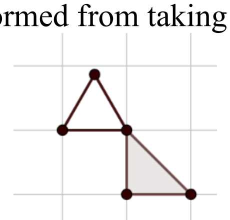



Part A. What is the measure of segment $BD$?
Part B. What is the measure of segment $AC$?
Part C. If the measure of segment $BD$ is 14 units, what is the measure of segment $BC$?

*Instructional Task 2 (MTR.7.1)*

Part A. A company is requesting equilateral tiles to be made for their new office floor. If the height of the tile is approximately 10.4 inches, what is the length of the sides of the triangle?

Part B. The same company decides they also want to use half of a square with the side the same length as the height of the equilateral triangle. What is the length of the hypotenuse of the triangle formed from taking half of the square?



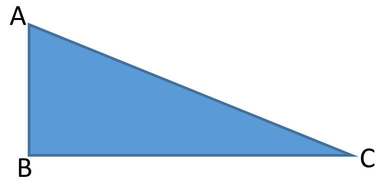
## Instructional Items

*Instructional Item 1*

The logo of a local construction company contains an equilateral triangle. The height of the triangle is 10 units. What is the length of the measure of each side of the triangle?

*Instructional Item 2*

The right triangle $ABC$ is shown. Angle $B$ is the right angle and the length of $AB$ is 1.5 centimeters and the length of $BC$ is 3.1 centimeters.




Part A. Determine the measure of angles $A$ and $C$.
Part B. Determine the length of $AC$.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

*MA.912.T.1.3*

## Benchmark

| | |
|---|---|
| **MA.912.T.1.3** | Apply the Law of Sines and the Law of Cosines to solve mathematical and real-world problems involving triangles. |

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.1.2, MA.912.GR.1.6
- MA.912.GR.6.3
- MA.912.GR.7.2, MA.912.GR.7.3

## Terms from the K-12 Glossary

- Angle
- Triangle

## Vertical Alignment

**Previous Benchmarks**

- MA.8.GR.1.1, MA.8.GR.1.2
- MA.8.GR.2.4

**Next Benchmarks**

- MA.912.T.1.5, MA.912.T.1.6, MA.912.T.1.7, MA.912.T.1.8
- MA.912.NSO.3.5, MA.912.NSO.3.9

## Purpose and Instructional Strategies

In grade 8, students solved problems involving triangles using Pythagorean Theorem, Triangle Sum Theorem and Triangle Inequality Theorem. In Geometry, students apply the Law of Sines and the Law of Cosines to solve problems involving right and non-right triangles. In later courses, students will use other trigonometric relationships and identities to solve real-world and mathematical problems involving sides and angles of triangles. Additionally, students will use knowledge of the Law of Sines and Law of Cosines when learning about vectors.

- Within the Geometry course, the expectation is to use angle measures given in degrees and not in radians. Additionally, it is not the expectation for students to master the trigonometric ratios of secant, cosecant and cotangent within this course.

- It is customary to use Greek letters to represent angle measures (e.g., $\theta, \alpha, \beta, \gamma$).
- The Law of Sines states that given a triangle with side lengths $a$, $b$ and $c$, and angle measures $\alpha$, $\beta$ and $\gamma$ (opposite to the sides respectively), then $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$ or $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$.
- Instruction includes making the connection between the Law of Sines and the congruence criteria of Angle-Side-Angle and Angle-Angle-Side.
  - For example, when given two angle measures and the included side length, students can use the Law of Sines to determine the missing side lengths and angle measures and justify that there is one unique triangle as indicated by the Angle-Side-Angle congruence criterion.
- Instruction includes making the connection between the Law of Sines and the Side-Side-Angle criterion, which is referred to as the ambiguous case.
  - For example, when given two side lengths and an angle measure of an opposite angle students can use the Law of Sines to determine the value of the sines of the missing angle measures. If the value of either of the sines is greater than one, then no triangle exists with those angle measures. If the value of both of the sines is less than one, then there are two possible triangles, one acute and one obtuse. If the value of either of the sines is equal to one, then there is exactly one triangle, a right triangle, with those measures.
- The Law of Cosines states that given a triangle with side lengths $a$, $b$ and $c$, and angle measures $\alpha$, $\beta$ and $\gamma$ (opposite to the sides respectively), then $a^2 = b^2 + c^2 - 2bc\cos\alpha$, $b^2 = a^2 + c^2 - 2ac\cos\beta$ and $c^2 = a^2 + b^2 - 2ab\cos\gamma$.
- Instruction includes making the connection between the Law of Cosines and the congruence criteria of Side-Side-Side and Side-Angle-Side.
  - For example, when given three side lengths, students can use the three versions of the Law of Cosines to determine the missing angle measures and justify that there is one unique triangle as indicated by the Side-Side-Side congruence criterion. However, if the side lengths do not satisfy the Triangle Inequality Theorem, then no triangle exists.
- Instruction includes making the connection between the Law of Cosines and the Side-Side-Angle criterion, which is referred to as the ambiguous case.
  - For example, when given two side lengths and an angle measure of an opposite angle students can use one of the versions of the Law of Cosines to obtain a quadratic equation for the missing side length. If there is no real solutions, then no triangle exists. If there are two real solutions, then there are two possible triangles, one acute and one obtuse. If there is exactly one real solution, then there is exactly one triangle, a right triangle.
- Instruction includes the understanding that the Pythagorean Theorem is a special case of the Law of Cosines when the included angle measure is 90°, whose cosine is zero. (*MTR.5.1*)
- Instruction includes the understanding that sines and cosines of obtuse angles are necessary when solving problems involving obtuse triangles. Enrichment of this benchmark includes the connection to the unit circle.

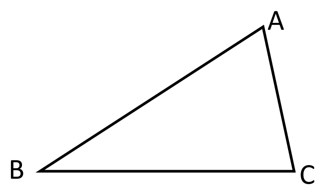
### Common Misconceptions or Errors

- Students may have trouble keeping track of which sides or angles belong in the Laws of Sines and Cosines.

## Instructional Tasks

*Instructional Task 1 (MTR.3.1, MTR.4.1)*

Triangle $ABC$ is an acute triangle. The measure of $AB$ is 17.8 inches, the measure of $BC$ is 16.5 inches and the measure of angle $C$ is 75°.




Part A. Determine the measure of angle $A$. Compare your method used with a partner.
Part B. Determine the measure of angle $B$.

*Instructional Task 2 (MTR.4.1, MTR.5.1)*

Part A. Given side lengths 10, 11 and 22, use the Law of Cosines to determine if a triangle exists with those side lengths.

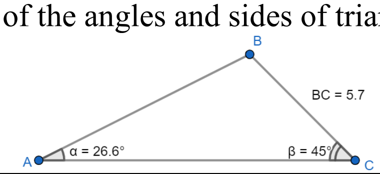
Part B. How could you determine, without using the Law of Cosines, whether such a triangle exists?

Part C. If the triangle from Part A does not exist, change one of its side lengths so that a triangle does exist.

## Instructional Items

*Instructional Item 1*

Find the remaining measures of the angles and sides of triangle $ABC$.




**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

*MA.912.T.1.4*

## Benchmark

**MA.912.T.1.4** Solve mathematical problems involving finding the area of a triangle given two sides and the included angle.

Benchmark Clarifications:

*Clarification 1:* Problems include right triangles, heights inside of a triangle and heights outside of a triangle.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.3.4
- MA.912.GR.4.3, MA.912.GR.4.4, MA.912.GR.4.6

## Terms from the K-12 Glossary

- Angle
- Area
- Hypotenuse
- Right Triangle

## Vertical Alignment

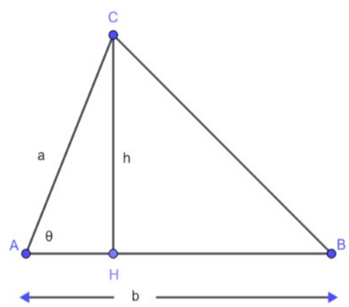
**Previous Benchmarks**
- MA.6.GR.2.1

**Next Benchmarks**
- MA.912.T.1.5, MA.912.T.1.6, MA.912.T.1.7, MA.912.T.1.8

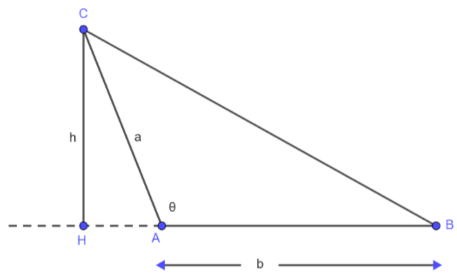
## Purpose and Instructional Strategies

In grade 6, students learned how to find the area of a triangle using a formula and its connection to the area of a rectangle. In Geometry, students find the area of a triangle when given the length of two sides and the included angle. In later courses, students will use further trigonometric relationships and identities to determine information that is needed to find areas of triangles.

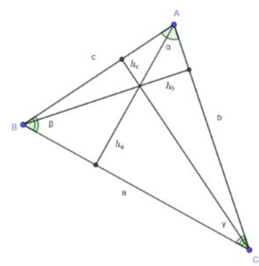
- Within the Geometry course, the expectation is to use angle measures given in degrees and not in radians. Additionally, it is not the expectation for students to master the trigonometric ratios of secant, cosecant and cotangent within this course.
- It is customary to use Greek letters to represent angle measures (e.g., $\theta, \alpha, \beta, \gamma$).
- Instruction includes making the connection between the formula for the area of a triangle ($A = \frac{1}{2}bh$) and the trigonometric functions (*MTR.5.1*).
  - For example, if an acute triangle is given with two sides lengths, $a$ and $b$, and the included angle measure, $\theta$, students can use trigonometric functions to determine the height of the triangle. Students can select side $b$ to be the base and then construct the height of the triangle, $h$, to determine the area. Students should realize that when constructing the height, it creates a right triangle and the length of $h$ can be found by using the sine of the angle $\theta$: $h = a\sin\theta$. Students can now use the area formula $A = \frac{1}{2}bh$ and substitution to obtain the new formula $A = \frac{1}{2}b(a\sin\theta)$. Similarly, students can use the side $a$ as the base and obtain the formula $A = \frac{1}{2}a(b\sin\theta)$.




  - For example, if an obtuse triangle is given with side lengths, $a$ and $b$, and the included obtuse angle measure, $\theta$, students can use trigonometric functions to determine the height of the triangle. Students can select side $b$ to be the base and then construct the height of the triangle, $h$, to determine the area. Students should realize that when constructing the height, it creates a right triangle outside of the given obtuse triangle and the length of $h$ can be found by using the sine of the angle $180 - \theta$: $h = a\sin(180 - \theta)$. Students can explore using technology or the unit circle to show that the $\sin(180 - \theta)$ is equivalent to $\sin(\theta)$, therefore $h = a\sin(\theta)$. Students can now use the area formula $A = \frac{1}{2}bh$ and substitution to obtain the new formula $A = \frac{1}{2}b(a\sin\theta)$. Similarly, students can use the side $a$ as the base and obtain the formula $A = \frac{1}{2}a(b\sin\theta)$.




- Instruction includes making the connection to the Law of Sines. (*MTR.5.1*)



  - For example, if given a triangle with side lengths a, b and c, and opposite angles $\alpha$, $\beta$ and $\gamma$, then there are three formulas students could use to determine the area: $A = \frac{1}{2}cb\sin\alpha$, $A = \frac{1}{2}ac\sin\beta$ and $A = \frac{1}{2}ba\sin\gamma$. Students should realize that each of these formulas are equivalent to one another, so that $\frac{1}{2}cb\sin\alpha = \frac{1}{2}ac\sin\beta = \frac{1}{2}ba\sin\gamma$. To make the connection to the Law of Sines, each expression can be divided by $\frac{1}{2}abc$ (Division Property of Equality), to obtain $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$.

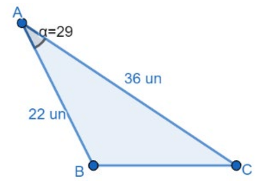
## Common Misconceptions or Errors

- Students may misidentify a side as the height of the triangle and attempt to use $A = \frac{bh}{2}$ to find the area.
- Students may have difficulty determining the height of an obtuse triangle.

## Instructional Tasks

*Instructional Task 1 (MTR.3.1, MTR.4.1)*

Triangle $ABC$ is given with $\overline{AB} = 22$ units, $\overline{AC} = 36$ units and the measure of angle $\alpha = 29°$.




Part A. If side $AB$ is the considered the base of the triangle, determine the height of the triangle.

Part B. If side $AC$ is the considered the base of the triangle, determine the height of the triangle.

Part C. Determine the area of the triangle. Compare your method with a partner.

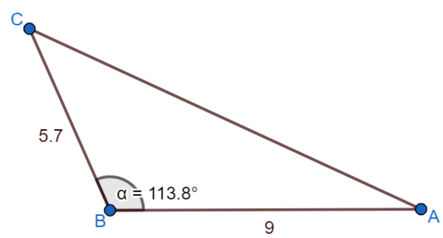
Part D. Determine the length of side $BC$ using the Law of Cosines.

Part E. Can you use the answers from Part C and Part D to determine the height of the triangle if $BC$ is considered as the base.

## Instructional Items

*Instructional Item 1*

Find the area of triangle $ABC$.




**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

**MA.912.LT.4** *Develop an understanding of the fundamentals of propositional logic, arguments and methods of proof.*

*MA.912.LT.4.3*

## Benchmark

**MA.912.LT.4.3** Identify and accurately interpret "if…then," "if and only if," "all" and "not" statements. Find the converse, inverse and contrapositive of a statement.

Benchmark Clarifications:
*Clarification 1:* Instruction focuses on recognizing the relationships between an "if…then" statement and the converse, inverse and contrapositive of that statement.
*Clarification 2:* Within the Geometry course, instruction focuses on the connection to proofs within the course.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.1
- MA.912.GR.2.6, MA.912.GR.2.7, MA.912.GR.2.8, MA.912.GR.2.9
- MA.912.GR.5

## Terms from the K-12 Glossary

## Vertical Alignment

**Previous Benchmarks**
- MA.8.GR.1

**Next Benchmarks**
- MA.912.LT.4
- MA.912.LT.5

## Purpose and Instructional Strategies

In grade 8, students were first introduced to theorems and their converses with the Triangle Sum, Triangle Inequality and Pythagorean Theorems and used, informally, logical statements about the angles formed by crossing lines and angles in polygons. In Geometry, students use and think more formally about different kinds of logical statements. In later courses, students refine their knowledge and skills regarding logic and set theory.

- Instruction of this benchmark should be done throughout the course, as students are using postulates, proving relationships and theorems and studying definitions.
- Instruction includes the student understanding that a good definition is precise and accurate, and can always be written in the "if and only if" form. Having this understanding helps students to avoid misconceptions within definitions.
  - For example, the definition of a parallelogram states that it is a quadrilateral with opposite sides parallel. This can be written as "a quadrilateral is a parallelogram if and only if its opposite sides are parallel." This means that "if a quadrilateral is a parallelogram, then it has opposite sides that are parallel," and that "if a quadrilateral has opposite sides that are parallel, then it is a parallelogram."
- Instruction includes discussing the meaning of "all" within statements.
  - For example, students can discuss whether it is true that *all* translations are rigid motions or whether it is true that *all* rigid motions are translations.
- Instruction includes the student understanding that a conditional statement has a hypothesis and a conclusion in the form of "if…then," where the *if* part of the statement

is the hypothesis and the *then* part of the statement is the conclusion. A conditional statement can be transformed by negating or rearranging its parts to create an inverse statement, a converse statement, a contrapositive statement and an "if and only if" statement.

- Conditional Statement
  When a statement is in form of "if…then," where the *if* part of the statement is the hypothesis and the *then* part of the statement is the conclusion.
  For example, given the conditional statement "If two angles are vertical, then the angles are congruent," the hypothesis is "two angles are vertical" and the conclusion is "the angles are congruent."
- Negation
  When a statement uses the word "not."
  For example, "two angles are not vertical" is the negation of the hypothesis and "the angles are not congruent" is the negation of the conclusion.
- Inverse Statement
  When both the hypothesis and conclusion of a conditional statement are negated.
  For example, "If two angles are not vertical, then the angles are not congruent."
- Converse Statement
  When the hypothesis and the conclusion are switched.
  For example, "If two angles are congruent, then the angles are vertical."
- Contrapositive Statement
  When both the hypothesis and conclusion of a conditional statement are negated and switched.
  For example, "If two angles are not congruent, then the angles are not vertical."
- "If and only if" Statement
  When a conditional statement and its converse are combined in an abbreviated way.
  For example, "Two angles are congruent, if and only if the two angles are vertical," is a compact way of saying "If two angles are vertical, then the angles are congruent, and if two angles are congruent, then the angles are vertical."

- Instruction includes interpreting conditional statements and negations or rearrangements of their hypothesis and conclusion and analyzing whether the resulting statements are true or false. Students can use postulates, definitions and proofs to determine whether a statement is true. Students can use counterexamples to show that a statement is false.
  - Given a true conditional statement, its converse statement or inverse statement may or may not be true.
    For example, given the true conditional statement "If an angle measure is 34°, then the angle is acute," its converse or inverse would not produce a true statement.
    For example, given the true conditional statement "If a triangle has side lengths $a$, $b$ and $c$ satisfying $a^2 + b^2 = c^2$, then the triangle is a right triangle," its converse would produce a true statement.
  - Given a true conditional statement, its contrapositive statement is also true.
    For example, given the true conditional statement "If an angle measure is 34°, then the angle is acute," its contrapositive would produce a true statement.
- For enrichment of this benchmark, instruction includes the use of truth tables to help

students organize their work.

## Common Misconceptions or Errors

- Students may confuse the hypothesis and conclusion.
- Students may try to change the inverse, converse or contrapositive to make sense in the real world rather than using the logic definitions.

## Instructional Tasks

*Instructional Task 1 (MTR.7.1)*

Use the statements below to identify the converse, inverse and contrapositive of the statement "If I can run a 5K race in under 27 minutes, then I can start the race at the front of the pack."

- If I cannot run a 5K race in under 27 minutes, then I can start the race at the front of the pack.
- If I cannot run a 5K race in under 27 minutes, then I cannot start the race at the front of the pack.
- If I can run a 5K race in under 27 minutes, then I cannot start the race at the front of the pack.
- If I cannot start the race at the front of the pack, then I cannot run a 5K race in under 27 minutes.
- If I can start the race at the front of the pack, then I can run a 5K race in under 27 minutes.

*Instructional Task 2 (MTR.4.1)*

Part A. Write an "if…then" statement involving a quadrilateral.

Part B. Rewrite the statement as an "if and only if" statement. How are the two statements different in their meaning?

## Instructional Items

*Instructional Item 1*

Use the following statement to answer the questions.

*A triangle is an equilateral triangle if and only if the triangle has three congruent sides.*

Part A. Write the two "if…then" statements that can be written from the given statement.

Part B. Write the converse of one of the conditional statements created in Part A.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

*MA.912.LT.4.8*

## Benchmark

**MA.912.LT.4.8** Construct proofs, including proofs by contradiction.

Benchmark Clarifications:
*Clarification 1:* Within the Geometry course, proofs are limited to geometric statements within the course.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.1
- MA.912.GR.2.6, MA.912.GR.2.7, MA.912.GR.2.8, MA.912.GR.2.9
- MA.912.GR.5

## Terms from the K-12 Glossary

## Vertical Alignment

**Previous Benchmarks**

- MA.7.GR.1
- MA.8.GR.1

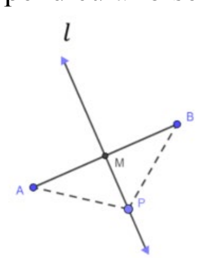
**Next Benchmarks**

- MA.912.LT.4
- MA.912.LT.5

## Purpose and Instructional Strategies

In grades 7 and 8, students explored the reasons why some geometric statements concerning angles and polygons are true or false. In Geometry, students learn and construct proofs for many of the geometric facts that they encounter. The content of this benchmark is to be used throughout this course. In later courses, students continue to learn and construct proofs in many different areas.

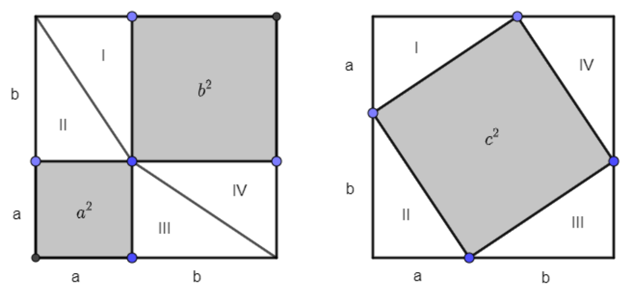
- Instruction includes the student understanding that proofs and proofs by contradiction can be represented in various ways. Students should have practice with each type of proof, understanding when one may be a more effective way to present information. *(MTR.2.1)*
    - Two-column proofs
      Organize the reasoning in two columns: statements and reasons. Each statement has a corresponding reason, and the proof usually starts with the given information.
      For example, given the perpendicular bisector of $\overline{AB}$, $l$, prove: $AP = BP$



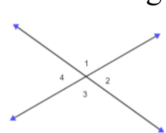

| Statements | Reasons |
|---|---|
| $l$ is the perpendicular bisector of $\overline{AB}$ | Given |
| $l \perp \overline{AB}$ and $M$ is the midpoint of $\overline{AB}$ | Definition of Perpendicular Bisector |

| | |
|---|---|
| $\overline{AM} \cong \overline{BM}$ | Definition of Midpoint |
| $\angle AMP$ and $\angle BMP$ are right angles | Definition of Perpendicular |
| $\angle AMP \cong \angle BMP$ | All Right Angles are Congruent |
| $\overline{MP} \cong \overline{MP}$ | Reflexive Property |
| $\Delta AMP \cong \Delta BMP$ | SAS Congruence Criterion |
| $\overline{AP} \cong \overline{BP}$ | Corresponding Parts of Congruent Triangles are Congruent (CPCTC) |
| $AP = BP$ | Definition of Congruent |

- Pictorial proofs
  Visual way to express the proof in its entirety. The picture can be accompanied by an explanation to provide background information or indicate the reasoning depicted by the picture.
  For example, the proof of the Pythagorean Theorem can be expressed as shown below.




  Both squares have the same side length, $a + b$, so they have the same area. The white region in each square consists of four right triangles ($I$, $II$, $III$ and $IV$), therefore the area of the white region in both squares is the same. Students can conclude the area of the grey region in each square is also the same showing that $a^2 + b^2 = c^2$.
- Paragraph and narrative proofs
  Consists of a logical argument written as a paragraph, giving evidence and detailed reasons to draw a conclusion. Paragraph proofs can be seen as a two-column proof written in sentences.
- Flow chart proofs
  A way to organize statements and reasons needed in a structured way to indicate the logical order. Statements are placed in boxes, reasons are placed under the box, and arrows are used to represent the flow or progression of the argument. For example, the proof that vertical angles are congruent is shown below.




- Informal proofs
  A way to provide convincing evidence to show that something is true. Informal proofs include the use of manipulatives, drawings and geometric software.

- Instruction includes the understanding that when a proof cannot be proved directly, it may be able to be proved by contradiction. A proof by contradiction assumes that the statement to be proved is not true and then uses a logical argument to deduce a contradiction. The logical argument can be represented in any form of a proof: two-column, pictorial, paragraph and flow chart.
  - For example, if students want to prove by contradiction that $\overline{BC} \ncong \overline{ST}$ given two triangles $ABC$ and $RST$, with $\overline{AC} \cong \overline{RT}$, $\overline{AB} \cong \overline{RS}$ and $\angle A \ncong \angle R$, they can start by assuming that $\overline{BC} \cong \overline{ST}$. Under this assumption, $\Delta ABC \cong \Delta RST$ by Side-Side-Side since $\overline{AC} \cong \overline{RT}$ and $\overline{AB} \cong \overline{RS}$ were given. Since the two triangle are congruent and corresponding parts of congruent triangles are congruent (CPCTC), then students can conclude that $\angle A \cong \angle R$. Students should realize that this contradicts the given information, $\angle A \ncong \angle R$. Therefore, this contradiction shows that the statement $\overline{BC} \cong \overline{ST}$ is false, proving that $\overline{BC} \ncong \overline{ST}$ is true.

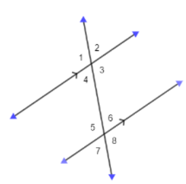
## Common Misconceptions or Errors

- Students may determine the statement of contradiction incorrectly.
  - For example, when completing a proof by contradiction, a student may create the statement of contradiction from one of the given pieces of information rather than what is to be proven.

## Instructional Tasks

*Instructional Task 1 (MTR.2.1, MTR.4.1)*

A pair of parallel lines is cut by a transversal as shown.




Part A. Using the Linear Pair Postulate and postulates involving parallel lines, prove that angle 1 is congruent to angle 8.

Part B. Compare your proof with a partner.

*Instructional Task 2 (MTR.3.1)*

Order the following statements to prove by contradiction that a triangle can only have one right angle.

1. The measure of angle $K$ is 90˚.
2. The measure of an angle in a triangle cannot equal 0˚.
3. In triangle $JKL$, only one of the angles can be a right angle.
4. $m\angle J + m\angle K + m\angle L = 180°$
5. Assume triangle $JKL$ has two right angles, angle $J$ and angle $K$.
6. The measure of angle J is 90˚.
7. $90° + 90° + m\angle L = 180°$
8. A triangle cannot have more than one right angle.
9. $m\angle L = 0°$
10. The sum of the measures of the angles in a triangle is 180˚.

## Instructional Items

*Instructional Item 1*

Use a proof by contradiction to prove the following statement.

*An equilateral triangle cannot also be a right triangle.*

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*

*MA.912.LT.4.10*

## Benchmark

**MA.912.LT.4.10** Judge the validity of arguments and give counterexamples to disprove statements.

Benchmark Clarifications:

*Clarification 1:* Within the Geometry course, instruction focuses on the connection to proofs within the course.

## Connecting Benchmarks/Horizontal Alignment

- MA.912.GR.1
- MA.912.GR.2.6, MA.912.GR.2.7, MA.912.GR.2.8, MA.912.GR.2.9
- MA.912.GR.5

## Terms from the K-12 Glossary

## Vertical Alignment

**Previous Benchmarks**

- MA.7.GR.1
- MA.8.GR.1

**Next Benchmarks**

- MA.912.LT.4
- MA.912.LT.5

## Purpose and Instructional Strategies

In grades 7 and 8, students explored the reasons why some geometric statements concerning angles and polygons are true or false. In Geometry, students learn and construct proofs for many of the geometric facts that they encounter and they learn to use counterexamples to check the validity of arguments and statements. The content of this benchmark is to be used throughout this course. In later courses, students continue to learn to judge the validity of many different arguments and statements.

- Instruction includes the understanding that a valid argument can be a statement or sequence of statements supported by valid reasons, a part of a proof or an entire proof.
- One way to show that an argument is not valid is to provide at least one counterexample to at least one statement in the argument.
    - For example, if the argument is "All rectangles have opposite sides parallel; therefore, given a quadrilateral is not a rectangle, the quadrilateral does not have opposite sides parallel," then a student can provide a parallelogram as a counterexample to show that concluding statement of the argument is not valid.

## Common Misconceptions or Errors

- Students may think a statement is true because they cannot think of a counterexample.

## Instructional Tasks

*Instructional Task 1 (MTR.3.1)*

Part A. Which of the following statements are true?

- If a quadrilateral is a square, then it is a rectangle.
- All trapezoids are parallelograms.
- Any quadrilateral can be inscribed in a circle.

Part B. Provide counterexamples to prove the invalid statements from Part A are not true.

## Instructional Items

*Instructional Item 1*

Puaglo said the following statements are true. Select all the statements that are false.

a. All quadrilaterals have four right angles.
b. A triangle is a polygon with three sides.
c. All circles are similar.
d. All equiangular quadrilaterals are congruent.
e. A trapezoid must have at least one obtuse angle.

**The strategies, tasks and items included in the B1G-M are examples and should not be considered comprehensive.*